ON GENERALIZED HILBERT ALGEBRAS

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Abstract. In this paper by considering the notion of generalized Hilbert algebra which is named g-Hilbert algebra, we obtain some properties of it. Moreover, we show that for all $n \geq 3$ there exist at least one proper g-Hilbert algebra of order n. Because g-Hilbert algebra is not a Boolean algebra we define the concept of branch in g-Hilbert algebras and we prove that any branch in commutative g-Hilbert algebras is a Boolean algebra.

Keywords: generalized Hilbert algebra, implication algebras, complemented lattice, distributive lattice, Boolean algebra.

AMS Subject Classification (2000): 06F35, 03G25, 08A30.

1. Introduction

Hilbert algebras [7] represent the algebraic counterpart of the implicative fragment of Intuitionistic Propositional Logic. In [7] Diego gives a topological representation for Hilbert algebras and he proves that every Hilbert algebra is isomorphic to a subalgebra of the implicative reduct of a Heyting algebra generated by a certain topological space. Also, Hilbert algebras, or positive implication algebras [14], are the duals of Henkin algebras called by him implicative models in [9]. Positive implicative BCK-algebras [11] are actually another version of Henkin algebras. As a matter of fact, these algebras are an algebraic counterpart of positive implicational calculus. Various expansions of Hilbert algebras by a conjunction-like operation have also been studied in the literature. The most extensively investigated among them are implicative semilattices, which are known also as Brouwerian semilattices.

Now, in this paper we give a generalization of positive implicative BCK-algebras and Hilbert algebras which is called a generalized Hilbert algebra that it is in form of variety. In follow, we obtain some properties of generalized Hilbert algebra and we show that any branch in commutative generalized Hilbert algebras is a Boolean algebra.

2. Generalized Hilbert algebras

Definition 2.1. [7] A *Hilbert algebra* is a triplet $(H, \to, 1)$ of type (2,0), where H is a nonempty set, "$\to$" is a binary operation which satisfies the following axioms:

(H1) $x \to (y \to x) = 1$,

(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

(H3) $x \to y = 1$ and $y \to x = 1$ imply $x = y$,

for all $x, y, z \in H$.

Proposition 2.2. *[8] If $(H, \to, 1)$ be a Hilbert algebra, then,*

(i) $x \to x = 1$,

(ii) $1 \to x = x$,

(iii) $x \to 1 = 1$,

(iv) $x \to (y \to z) = y \to (x \to z)$,

(v) $x \to (y \to z) = (x \to y) \to (x \to z)$,

for all $x, y, z \in H$.

Definition 2.3. A *generalized Hilbert algebra* (or briefly, *g-Hilbert algebra*) is an algebra $(G_H, \to, 1)$ of type (2,0) which satisfies the following axioms;

(GH1) $1 \to x = x$,

(GH2) $x \to x = 1$,

(GH3) $z \to (y \to x) = y \to (z \to x)$,

(GH4) $z \to (y \to x) = (z \to y) \to (z \to x)$,

for all $x, y, z \in G_H$.

Example 2.4. Let $(X, \leq, 1)$ be a unital poset and implication " $\to$ " on X is defined as follows:

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Then $(X, \to, 1)$ is a g-Hilbert algebra.

Example 2.5. Let $(X, \leq)$ be a poset. Then $Y \subseteq X$ is called *increasing subset* if it is closed under $\leq$, i.e for every $x \in Y$ and every $y \in X$ if $x \leq y$ then $y \in Y$. Now, let $\mathcal{P}_i(X)$ be the set of all increasing subset of X and for any $y \in X$, $[y) = \{x \in X : y \leq x\}$. Then it is easy to see that $(\mathcal{P}_i(X), \rightarrow, X)$ is a g-Hilbert algebra where the implication " $\rightarrow$ " is defined by

$$U \rightarrow V = \{x \in X : [x) \cap U \subseteq V\}$$

for $U, V \in \mathcal{P}_i(X)$.

Theorem 2.6. *Any Hilbert algebra is a g-Hilbert algebra.*

Proof. The proof is clear by Proposition 2.2. ■

Note. The converse of Theorem 2.6 is not correct in general.

Example 2.7. Let G_H={a,b,1} and operation $\rightarrow$ on G_H is defined as follows

$\rightarrow$	a	b	1
a	1	1	1
b	1	1	1
1	a	b	1

It is routine to check G_H={a,b,1} is a g-Hilbert algebra but it is not a Hilbert algebra, since $a \rightarrow b = b \rightarrow a = 1$ but $a \neq b$.

Proposition 2.8. *Let* $(G_H, \rightarrow, 1)$ *be a g-Hilbert algebra. Then:*

(i) $x \rightarrow 1 = 1$,

(ii) $(y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow (y \rightarrow x)) = 1$,

(iii) $(z \rightarrow x) \rightarrow ((y \rightarrow z) \rightarrow (y \rightarrow x)) = 1$,

(iv) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(v) $x \rightarrow (x \rightarrow y) = x \rightarrow y$,

(vi) $x \rightarrow (y \rightarrow x) = 1$,

(vii) $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$.

for all $x, y, z \in G_H$.

Proof. (i) Let $x \in G_H$. Then, by (GH2) and (GH4),

$$1 = 1 \rightarrow 1 = (x \rightarrow x) \rightarrow (x \rightarrow x) = x \rightarrow (x \rightarrow x) = x \rightarrow 1.$$

(ii) Let $x, y, z \in G_H$. Then,

$$\begin{aligned}(y \to z) \to & ((z \to x) \to (y \to x)) \\ & = (z \to x) \to ((y \to z) \to (y \to x)), \quad \text{(by (GH3))} \\ & = (z \to x) \to (y \to (z \to x)), \quad \text{(by (GH4))} \\ & = y \to ((z \to x) \to (z \to x)), \quad \text{(by (GH3))} \\ & = y \to 1, \quad \text{(by (GH2))} \\ & = 1 \quad \text{(by (i))}.\end{aligned}$$

(iii) Let $x, y, z \in G_H$. Then by (GH3) and (ii);

$$(z \to x) \to ((y \to z) \to (y \to x)) = (y \to z) \to ((z \to x) \to (y \to x)) = 1.$$

(iv) Let $x, y, z \in G_H$. Then by (GH4) and (GH2);

$$\begin{aligned}(x \to (y \to z)) \to & ((x \to y) \to (x \to z)) \\ & = ((x \to y) \to (x \to z)) \to ((x \to y) \to (x \to z)) = 1.\end{aligned}$$

(v) Let $x, y, z \in G_H$. Then by (GH4) and (GH1);

$$x \to (x \to y) = (x \to x) \to (x \to y) = 1 \to (x \to y) = x \to y.$$

(vi) Let $x, y, z \in G_H$. Then by (GH4) and (i);

$$x \to (y \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1.$$

(vii) Let $x, y, z \in G_H$. Then by (GH3) and (GH2);

$$y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1. \qquad \blacksquare$$

Definition 2.9. Let $(G_H, \to, 1)$ be a g-Hilbert algebra, then, G_H is called a *proper g-Hilbert algebra* if it is not a Hilbert algebra.

Proposition 2.10. *If G_H is a proper g-Hilbert algebra of order n, then $n \geq 3$.*

Proof. By Definition 2.3, Proposition 2.8 and Example 2.7, the proof is clear. $\blacksquare$

Theorem 2.11. *Let $(G_H, \to, 1)$ be a proper g-Hilbert algebra and $a \notin G_H$. Then $G'_H = G_H \cup \{a\}$ with the following operation is a proper g-Hilbert algebra.*

$$x \to y = \begin{cases} x \to y & , \quad x, y \in G_H, \\ a & , \quad x = 1, y = a, \\ 1 & , \quad x \in G'_H - \{1\}, y = a, \\ y & , \quad x = a, y \in G_H. \end{cases}$$

Proof. The proof of axioms (GH1), (GH2) and (GH3) are clear. So, we should only prove the axiom (GH4). For this case, we consider the following cases:

Case 1. $x, y \in G_H$ and $z = a$:

$$\begin{aligned} z \to (x \to y) &= a \to (x \to y) = x \to y \\ &= (a \to x) \to (a \to y) = (z \to x) \to (z \to y). \end{aligned}$$

Case 2. $x, z \in G_H$ and $y = a$:
If $x \neq 1$ and $z \neq 1$, then

$$\begin{aligned} z \to (x \to y) &= z \to (x \to a) = 1 = (z \to x) \to 1 \\ &= (z \to x) \to (z \to a) = (z \to x) \to (z \to y). \end{aligned}$$

If $x \neq 1$ and $z = 1$, then

$$\begin{aligned} z \to (x \to y) &= z \to (x \to a) = 1 = x \to a = x \to y \\ &= (1 \to x) \to (1 \to y) = (z \to x) \to (z \to y). \end{aligned}$$

If $x = 1$ and $z \neq 1$, then

$$z \to (x \to y) = z \to y = 1 \to (z \to y) = (z \to x) \to (z \to y).$$

If $x = 1$ and $z = 1$, then

$$z \to (x \to y) = y = 1 \to y = (1 \to 1) \to (1 \to y) = (z \to x) \to (z \to y).$$

Case 3. $y, z \in G_H$ and $x = a$.
The proof is similar to the proof of Case 2, by some modification.

Case 4. $x \in G_H$ and $y = z = a$.
If $x \neq 1$, then

$$\begin{aligned} z \to (x \to y) &= a \to (x \to y) = x \to y \\ &= (a \to x) \to (a \to y) = (z \to x) \to (z \to y). \end{aligned}$$

If $x = 1$, then

$$\begin{aligned} z \to (x \to y) &= a \to (1 \to a) = a \to a = 1 = 1 \to 1 \\ &= (a \to 1) \to (a \to a) = (z \to x) \to (z \to y). \end{aligned}$$

Case 5. $y \in G_H$ and $x = z = a$ or $z \in G$ and $y = x = a$:
The proof is similar to the proof of Case 4, by some modification.

Case 6. $x = y = z = a$:

$$\begin{aligned} z \to (x \to y) &= a \to (a \to a) = a \to 1 = 1 = 1 \to 1 \\ &= (a \to a) \to (a \to a) = (z \to x) \to (z \to y). \end{aligned}$$

Hence, $(G'_H, \to, 1)$ is a g-Hilbert algebra. ■

Corollary 2.12. *For any natural number $n \geq 3$, there exist at least one proper g-Hilbert algebra of order n.*

Definition 2.13. Let $(G_H, \to, 1)$ be a g-Hilbert algebra and $a \in G_H$. Then, the set $B(a) = \{x \in G_H | a \to x = 1\}$ is called a *branch* of X.

It is clear that $1, a \in B(a)$ and so $B(a) \neq \emptyset$.

Theorem 2.14. *Let $(G_H, \to, 1)$ be a g-Hilbert algebra such that for all $x, y \in G_H$, $B(x) \cap B(y) = \{1\}$ and $x \to y \neq y$. Then G_H is a proper g-Hilbert algebra.*

Proof. Let G_H be a Hilbert algebra, by contrary. By Proposition 2.2(iv) and (i), $y \to ((y \to x) \to x) = 1$ and so $(y \to x) \to x \in B(y)$. Now, let $z = y \to x$. Then, by Proposition 2.2(iv), (i) and (iii),

$$x \to (z \to x) = z \to (x \to x) = z \to 1 = 1$$

and so, $(y \to x) \to x = z \to x \in B(x)$. Hence,

$$(y \to x) \to x \in B(x) \cap B(y) = \{1\}$$

and so, $(y \to x) \to x = 1$. On the other hand, by (GH4) and (GH2) and Proposition 2.2(iii),

$$x \to (y \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1$$

and so, by (H3) we get that, $y \to x = x$, which is a contradiction. Therefore, G_H is a proper g-Hilbert algebra. ■

3. Generalized Hilbert algebra induced by a quasi ordered set

From now one in this paper, G_H denote a g-Hilbert algebra, unless otherwise mentioned.

Proposition 3.1. *Let relation $\preceq$ on G_H be defined as follows:*

$$x \preceq y \quad \textit{if and only if} \quad x \to y = 1$$

Then " $\preceq$ " is a quasi order relation.

Proof. Reflexive condition is clear. Now, we should prove the transitive condition. Let $x, y, z \in G_H$. If $x \preceq y$ and $y \preceq z$, then $x \to y = 1$ and $y \to z = 1$ and so by $(GH1)$ and $(GH4)$,

$$x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = x \to (y \to z) = x \to 1 = 1$$

Then $x \preceq z$. ■

Proposition 3.2. *Let $x \preceq y$, for $x, y \in G_H$. Then, for all $z \in G_H$,*

(i) $y \to z \preceq x \to z$,

(ii) $z \to x \preceq z \to y$.

Proof. (i) Since $x \to y = 1$, then

$$\begin{aligned}(y \to z) \to (x \to z) &= 1 \to ((y \to z) \to (x \to z))\\ &= (x \to y) \to ((y \to z) \to (x \to z))\\ &= (y \to z) \to ((x \to y) \to (x \to z))\\ &= (y \to z) \to (x \to (y \to z))\\ &= x \to ((y \to z) \to (y \to z))\\ &= x \to 1 = 1\end{aligned}$$

Hence, $y \to z \preceq x \to z$.

(ii) Since $x \to y = 1$, then by (GH4),

$$(z \to x) \to (z \to y) = z \to (x \to y) = z \to 1 = 1$$

Hence, $z \to x \preceq z \to y$. ∎

We define Θ on G_H as follows:

$$x\Theta y \Longleftrightarrow x \preceq y, y \preceq x$$

Then, Θ is a congruence relation on G_H. It is clear that Θ is an equivalence relation on G_H. Let $x, y, u, v \in G_H$, such that $x\Theta y$ and $u\Theta v$. Then $x \preceq y$, $y \preceq x$, $u \preceq v$ and $v \preceq u$. By Proposition 3.2, we obtain $x \to u \preceq x \to v$ and $x \to v \preceq y \to v$. Now, by transitivity of $\preceq$, we get $x \to u \preceq y \to v$. Similarly, we have $y \to v \preceq x \to u$ and so Θ is a congruence relation on G_H.

Now, let $\frac{G_H}{\Theta} = \{[x]_\Theta | x \in G_H\}$ and $\ll$ on $\frac{G_H}{\Theta}$ is defined as follows:

$$[x] \ll [y] \Longleftrightarrow x\Theta y.$$

It is clear that $(\frac{G_H}{\Theta}, \ll)$ is a poset.

Furthermore, $(\frac{G_H}{\Theta}, \ll, [1]_\Theta)$ is a g-Hilbert algebra with the following operation,

$$[x]_\Theta \to [y]_\Theta = [x \to y]_\Theta$$

Theorem 3.3. *Suppose that (P, θ) is a quasi ordered set, $1 \notin P$ and $G_H = P \cup \{1\}$. Let "$\to$" on G_H is defined as follows:*

$$x \to y = \begin{cases} 1 & , \quad x\theta y, \\ y & , \quad x \not\theta y. \end{cases}$$

Then $(G_H, \to, 1)$ is a g-Hilbert algebra.

Proof. Since θ is reflexive, obviously $x \to x = 1$, for all $x \in G_H$. Since $1 \notin P$, then $1 \not\theta x$ for every $x \in G_H$ and so $1 \to x = x$. Hence we have (GH1) and (GH2). Now, we should prove (GH3). Let $x, y, z \in G_H$. We consider the following cases:

Case 1. $y \not\theta x$ and $z \not\theta x$:

$$z \to (y \to x) = z \to x = x = y \to x = y \to (z \to x).$$

Case 2. $y\theta x$ and $z \not\theta x$:

$$z \to (y \to x) = z \to 1 = 1 = y \to x = y \to (z \to x).$$

Case 3. $y \not\theta x$ and $z\theta x$:

$$z \to (y \to x) = z \to x = 1 = y \to 1 = y \to (z \to x).$$

Case 4. $y\theta x$ and $z\theta x$:

$$z \to (y \to x) = z \to 1 = 1 = y \to 1 = y \to (z \to x).$$

Hence, we have (GH3).

Now, we will prove (GH4). Let $x, y, z \in G_H$. Then, we consider the following cases:

Case 1. $z\theta y$ and $z\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to 1 = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to 1 = 1 = z \to x = z \to (y \to x).$$

Case 2. $z \not\theta y$ and $z\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = y \to 1 = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = y \to 1 = 1 = z \to x = z \to (y \to x).$$

Case 3. $z\theta y$ and $z \not\theta x$:

If $x\theta y$, then by transitive condition $z\theta x$, which is not impossible.

If $x \not\theta y$, then:

$$(z \to y) \to (z \to x) = 1 \to x = x = z \to x = z \to (y \to x).$$

Case 4. $z \not\theta y$ and $z \not\theta x$:

If $x\theta y$, then:

$$(z \to y) \to (z \to x) = y \to x = 1 = z \to 1 = z \to (y \to x).$$

If $x \not\theta y$, then

$$(z \to y) \to (z \to x) = y \to x = x = z \to x = z \to (y \to x).$$

Hence, we have $(GH4)$. Therefore, G_H is a g-Hilbert algebra. ∎

4. Relation between generalized Hilbert algebras and implication algebras

Definition 4.1. [1] An *implication algebra* is a set X with a binary operation "$\to$" which satisfies the following axioms:

(I1) $(x \to y) \to x = x$,

(I2) $(x \to y) \to y = (y \to x) \to x$,

(I3) $x \to (y \to z) = y \to (x \to z)$,

for all $x, y, z \in X$.

In any implication algebra $(X, \to)$, we have

(I4) $x \to (x \to y) = x \to y$,

(I5) $x \to x = y \to y$,

(I6) there exists a unique element 1 in X such that, for all $x \in X$,

- (a) $x \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$,
- (b) if $x \to y = 1$ and $y \to x = 1$ then $x = y$,
- (c) $x \to (y \to x) = 1$

for all $x, y \in X$.

Definition 4.2. G_H is called *commutative* if for all $x, y \in G_H$,

$$(y \to x) \to x = (x \to y) \to y.$$

Lemma 4.3. *Let G_H be commutative. If $x \to y = y \to x = 1$, then $x = y$.*

Proof. Let $x \to y = y \to x = 1$, for $x, y \in G$. Since G_H is commutative, then by (GH1),

$$x = 1 \to x = (y \to x) \to x = (x \to y) \to y = 1 \to y = y.$$ ∎

Lemma 4.4. [1] *Let $(X, \to, 1)$ be an implication algebra. Then,*

(i) $x \leq y$, *imply* $y \to z \leq x \to z$

(ii) $x \leq y$, *imply* $z \to x \leq z \to y$

(iii) $x \to y \leq (y \to z) \to (x \to z)$ *and* $y \to z \leq (x \to y) \to (x \to z)$.

Theorem 4.5. $(X, \to, 1)$ *is an implication algebra if and only if* $(X, \to, 1)$ *is a commutative g-Hilbert algebra.*

Proof. $(\Rightarrow)$ Let $(X, \to, 1)$ be an implication algebra. By Theorem 2.6, it is enough to prove that X is a Hilbert algebra. By (I6)(b) and (c), we have (H1) and (H3). It is enough to prove (H2). Let $x, y, z \in X$. Then, by (I4), (I3) and Lemma 4.4,

$$\begin{aligned}
(x \to (y \to z)) &\to ((x \to y) \to (x \to z)) \\
&= (x \to (y \to z)) \to ((x \to y) \to (x \to (x \to z))), \\
&\geq (x \to (y \to z)) \to (y \to (x \to z))), \\
&= (x \to (y \to z)) \to (x \to (y \to z))), \\
&= 1.
\end{aligned}$$

Hence, by (I6)(a),(b), we have $(x \to (y \to z)) \to ((x \to y) \to (x \to z))) = 1$, and so (H2) is hold. Hence, $(X, \to, 1)$ is a Hilbert algebra and so, by Theorem 2.6, it is a g-Hilbert algebra. Moreover, by (I2) it is commutative.

$(\Leftarrow)$ Let $(X, \to, 1)$ be a commutative g-Hilbert algebra. Since X is commutative, then we have (I2). Moreover, by (GH3), we have (I3). Now, it is enough to prove that (I1). Let $x, y \in X$. Then, by (GH3), (GH2) and Proposition 2.8(i),

$$x \to ((x \to y) \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1$$

Hence,

$$x \to ((x \to y) \to x) = 1. \tag{1}$$

Moreover,

$$\begin{aligned}
((x \to y) \to x) \to x &= (x \to (x \to y)) \to (x \to y), && \text{Since } X \text{ is commutative} \\
&= ((x \to x) \to (x \to y)) \to (x \to y), && \text{by (GH4)} \\
&= (1 \to (x \to y)) \to (x \to y), && \text{by (GH2)} \\
&= (x \to y) \to (x \to y) \\
&= 1.
\end{aligned}$$

Hence,

$$((x \to y) \to x) \to x = 1, \tag{2}$$

and so, by (1), (2) and Lemma 4.3 we have (I1). Therefore, $(X, \to, 1)$ is an implication algebra. ∎

Example 4.6. Let $G_H = \{a, b, c, 1\}$ and operation "$\to$" on G_H is defined as follows:

$\to$	a	b	c	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
1	a	b	c	1

Then, $(G_H, \to, 1)$ is a g-Hilbert algebra which is not an implication algebra, since $(b \to c) \to c \neq (c \to b) \to b$. Hence, a commutative condition is necessary in the last theorem.

5. Lattice structure on commutative generalized Hilbert algebras

Proposition 5.1. *If G_H is commutative, then*

(i) $(x \to y) \to x = x$,

(ii) $x \to (x \to y) = x \to y$,

for all $x, y \in G_H$.

Proof. (i) Let $x, y \in G_H$. Then,

$$\begin{aligned} x \to ((x \to y) \to x) &= (x \to (x \to y)) \to (x \to x), \quad \text{(by (GH4))} \\ &= (x \to (x \to y)) \to 1, \quad \text{(by (GH2))} \\ &= 1 \end{aligned}$$

On the other hand,

$$\begin{aligned} ((x \to y) \to x) \to x &= (x \to (x \to y)) \to (x \to y), \quad \text{(since } G_H \text{ is commutative)} \\ &= ((x \to x) \to (x \to y)) \to (x \to y), \quad \text{(by (GH4))} \\ &= (1 \to (x \to y)) \to (x \to y), \quad \text{(by (GH2))} \\ &= (x \to y) \to (x \to y), \quad \text{(by (GH1))} \\ &= 1 \quad \text{(by (GH2))}. \end{aligned}$$

Hence, by Lemma 4.3 we obtain $(x \to y) \to x = x$.

(ii) By using (i) twice, we have

$$x \to (x \to y) = ((x \to y) \to x) \to (x \to y) = x \to y. \qquad \blacksquare$$

Corollary 5.2. *Let G_H be commutative and relation $\preceq$ on G_H, is defined by $x \preceq y$ if and only if $x \to y = 1$. Then, "$\preceq$" is a partial order on G_H.*

Proof. By (GH2), Proposition 5.1(i) we get that $\preceq$ is reflexive and anti symmetric. Let $x \to y = 1$ and $y \to z = 1$, then

$$x \to z = 1 \to (x \to z) = (x \to y) \to (x \to z) = x \to (y \to z) = x \to 1 = 1.$$

Thus, $\preceq$ is a partial order on G_H. $\blacksquare$

Proposition 5.3. *For any $p \in G_H$, $B(p)$ is a subalgebra of G_H.*

Proof. It is clear that $1 \in B(p)$. Now, let $a, b \in B(p)$. Then, $p \preceq a$ and $p \preceq b$ and so by (GH4)and (GH2),

$$p \to (a \to b) = (p \to a) \to (p \to b) = 1 \to 1 = 1.$$

Hence, $p \preceq (a \to b)$ and so $a \to b \in B(p)$. Therefore, $B(p)$ is a subalgebra of G_H. $\blacksquare$

Theorem 5.4. *If G_H is commutative, then the following are hold:*

(i) $(G_H, \vee)$ *is a* $\vee-$*semi lattice, when* $a \vee b = (a \to b) \to b$*, for any* $a, b \in G_H$*,*

(ii) *For any* $p \in G_H$*,* $(B(p), \wedge)$ *is a* $\wedge-$*semi lattice, when* $a \wedge b = ((a \to p) \vee (b \to p)) \to p$*, for any* $a, b \in B(p)$*,*

(iii) *For any* $p \in G_H$*,* $(B(p), \wedge, \vee)$ *is a complemented lattice.*

Proof. By Corollary 5.2, $(G_H, \preceq)$ and so $(B(p), \preceq)$, for any $p \in G_H$ is a partial ordered set. (i) Let $a, b \in G_H$. First we should prove that $(a \to b) \to b$ is an upper bound of a, b. By (GH3) and (GH2), $a \to ((a \to b) \to b) = (a \to b) \to (a \to b) = 1$, and so $a \preceq (a \to b) \to b$. Moreover, by (GH3), $b \preceq (a \to b) \to b$. Hence, $(a \to b) \to b$ is an upper bound of a, b. Now, let $c \in G$ such that $a, b \preceq c$. Then $a \to c = 1$ and so by commutative condition and (GH1),

$$(c \to a) \to a = (a \to c) \to c = 1 \to c = c. \tag{1}$$

Hence,

$$
\begin{aligned}
&((a \to b) \to b) \to c \\
&\quad = ((a \to b) \to b) \to ((c \to a) \to a), \quad \text{(by (1))} \\
&\quad = (c \to a) \to (((a \to b) \to b) \to a), \quad \text{(by (GH3))} \\
&\quad = (c \to a) \to (((b \to a) \to a) \to a), \quad \text{(by commutative condition)} \\
&\quad = (c \to a) \to ((a \to (b \to a)) \to (b \to a)), \quad \text{(by commutative condition)} \\
&\quad = (c \to a) \to ((b \to (a \to a)) \to (b \to a)), \quad \text{(by (GH3))} \\
&\quad = (c \to a) \to ((b \to 1) \to (b \to a)), \quad \text{(by (GH2))} \\
&\quad = (c \to a) \to (1 \to (b \to a)), \quad \text{(by Proposition 2.8(i))} \\
&\quad = (c \to a) \to (b \to a), \quad \text{(by } (GH1)\text{)} \\
&\quad = b \to ((c \to a) \to a), \quad \text{(by (GH3))} \\
&\quad = b \to c, \quad \text{(by (1))} \\
&\quad = 1 \quad \text{(since } b \preceq c\text{)}.
\end{aligned}
$$

Therefore, $(a \to b) \to b \preceq c$ and so $a \vee b = (a \to b) \to b$. Hence, $(G_H, \vee)$ is a $\vee-$semi lattice.

(ii) Let $p \in G_H$ and $a, b \in B(p)$. Then $p \preceq a, b$.
Since $a \to p \preceq (a \to p) \vee (b \to p)$ then, by Proposition 3.2,

$$((a \to p) \vee (b \to p)) \to p \preceq (a \to p) \to p = a \vee p = a.$$

Similarly, we can prove that $((a \to p) \vee (b \to p)) \to p \preceq b$. Hence, $((a \to p) \vee (b \to p)) \to p$ is a lower bound of a and b. Now, let $c \in G_H$ such that $c \preceq a, b$. Then, by Proposition 3.2, $a \to p \preceq c \to p$ and $b \to p \preceq c \to p$ and so $((a \to p) \vee (b \to p)) \preceq c \to p$. Hence, by Proposition 3.2, $c \preceq c \vee p = (c \to p) \to$

$p \preceq [a \to p) \vee (b \to p)] \to p$. Therefore, $((a \to p) \vee (b \to p)) \to p$ is a greatest lower bound of a and b and so

$$a \wedge b = ((a \to p) \vee (b \to p)) \to p.$$

Now, since $a \wedge b \in B(p)$, then $(B(p), \wedge)$ is a $\wedge-$semilattice.

(iii) Let $p \in G_H$. Then, for any $a \in B(p)$,

$$\begin{aligned} (a \to p) \vee a &= ((a \to p) \to a) \to a \\ &= (a \to (a \to p)) \to (a \to p), \quad \text{(by commutative condition)} \\ &= (a \to p) \to (a \to p), \quad \text{(by Proposition 2.8(v))} \\ &= 1, \quad \text{(by } (GH2)) \end{aligned}$$

Moreover, by commutative condition,

$$\begin{aligned} a \wedge (a \to p) &= ((a \to p) \vee ((a \to p) \to p)) \to p \\ &= ((a \to p) \vee ((p \to a) \to a)) \to p \\ &= ((a \to p) \vee (p \vee a)) \to p \\ &= (a \to p) \vee a) \to p \\ &= 1 \to p \\ &= p \end{aligned}$$

Therefore, $(B(p), \wedge, \vee)$ is a complemented lattice. ∎

Lemma 5.5. *Let G_H be a commutative g-Hilbert algebra and $p \in G_H$. Then, for any $a, b \in B(p)$,*

$$(a \to p) \vee (b \to p) = (a \wedge b) \to p.$$

Proof. Let $p \in G_H$ and $a, b \in B(p)$. Since $a \wedge b \preceq a$ and $a \wedge b \preceq b$, then, by Proposition 3.2, $a \to p \preceq (a \wedge b) \to p$ and $b \to p \preceq (a \wedge b) \to p$ and so $(a \wedge b) \to p$ is an upper bound of $a \to p$ and $b \to p$. Now, let $u \in B(p)$ be an other upper bound of $a \to p$ and $b \to p$. Then, $a \to p \preceq u$ and $b \to p \preceq u$ and so, by Proposition 3.2, $u \to p \preceq (a \to p) \to p$ and $u \to p \preceq (b \to p) \to p$. Hence, $u \to p \preceq a \vee p = a$ and $u \to p \preceq b \vee p = b$ and so $u \to p \preceq a \wedge b$. Thus,

$$a \wedge b \to p \preceq (u \to p) \to p = u \vee p = u$$

Therefore, $a \wedge b \to p$ is last upper bound of $a \to p$ and $b \to p$, that is,

$$(a \to p) \vee (b \to p) = (a \wedge b) \to p.$$ ∎

Theorem 5.6. *If G_H is commutative, then for any $p \in G_H$, $(B(p), \vee, \wedge)$ is a distributive lattice.*

Proof. Let $p \in G_H$ and $a, b, c \in B(p)$. We have to prove that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Since $p \preceq b$, then by Proposition 3.2, $a \to p \preceq a \to b$. Moreover, by Proposition 2.8(vi), $b \preceq a \to b$ and so $(a \to p) \vee b \preceq a \to b$. Let $c = (a \to p) \vee b$. Then

$$c \preceq a \to b \text{ and } b \preceq c \text{ and so } c = b \vee c = (c \to b) \to b. \tag{1}$$

Moreover, by the proof of Theorem 5.4(iii),

$$(a \to c) \to c = a \vee c = a \vee (a \to p) \vee b = 1 \vee b = 1. \tag{2}$$

Moreover,

$$\begin{aligned}
(a \to b) \to c &= (a \to b) \to (1 \to c), \quad \text{(by (GH1))} \\
&= (a \to b) \to (((a \to c) \to c) \to c), \quad \text{(by (2))} \\
&= (a \to b) \to ((a \to c) \vee c), \\
&= (a \to b) \to (a \to c), \quad \text{(by Proposition 2.8(vi))} \\
&= (a \to b) \to (a \to ((c \to b) \to b), \quad \text{(by (1))} \\
&= (a \to b) \to ((c \to b) \to (a \to b)), \quad \text{(by (GH3))} \\
&= (c \to b) \to ((a \to b) \to (a \to b)), \quad \text{(by (GH3))} \\
&= (c \to b) \to 1, \quad \text{(by (GH1))} \\
&= 1.
\end{aligned}$$

and this implies that $a \to b \preceq c$. Hence, by (1) and Proposition 5.1(i),

$$a \to b = c = (a \to p) \vee b. \tag{2}$$

Now, since (2) holds for any $a, b \in B(p)$ and since $B(p)$ is a subalgebra, then $a, b \to p \in B(p)$ and so, by (2),

$$(a \to (b \to p)) \to p = ((a \to p) \vee (b \to p)) \to p = a \wedge b. \tag{3}$$

Hence,

$$\begin{aligned}
a \to (a \wedge b) &= a \to ((a \to (b \to p)) \to p), \quad \text{(by (3))} \\
&= (a \to (b \to p)) \to (a \to p), \quad \text{(by Proposition 2.8(v))} \\
&= (b \to (a \to p)) \to (a \to p), \quad \text{(by (GH3))} \\
&= b \vee (a \to p), \\
&= a \to b, \quad \text{(by (2))}
\end{aligned}$$

and this implies that

$$a \to (a \wedge b) = a \to b. \tag{4}$$

Now, let $k = (a \wedge b) \vee (a \wedge c)$. Since, $a \wedge b \preceq k$, then by Proposition 3.2, (GH1), (GH2), (GH3) and (4),

$$1 = a \to 1 = a \to (b \to b) = b \to (a \to b) = b \to (a \to (a \wedge b)) \preceq b \to (a \to k)$$

and so, by Proposition 2.8(i), $b \to (a \to k) = 1$ and this implies that $b \preceq a \to k$. Similarly, $c \preceq a \to k$ and so

$$b \vee c \preceq a \to k. \tag{5}$$

and, by Proposition 3.2,

$$(a \to k) \to k \preceq (b \vee c) \to k. \tag{6}$$

Now, since $a \wedge b \preceq a$ and $a \wedge c \preceq a$, then $k = (a \wedge b) \vee (a \wedge c) \preceq a$ and so $k \to a = 1$. Hence, by (6) and the commutative condition,

$$a = 1 \to a = (k \to a) \to a = (a \to k) \to k \preceq (b \vee c) \to k. \tag{7}$$

Hence, by (5), (7) and the commutative property,

$$(a \to k) \vee ((b \vee c) \to k) \succeq (a \to k) \vee a = ((a \to k) \to a) \to a = a \to a = 1.$$

Now, since $(a \to k) \vee ((b \vee c) \to k) \preceq 1$, then, by Proposition 5.1(i),

$$(a \to k) \vee (b \vee c) \to k) = 1. \tag{8}$$

Moreover, since $a \wedge b \preceq b \vee c$ and $a \wedge c \preceq b \vee c$, then $k = (a \wedge b) \vee (a \wedge c) \preceq b \vee c$. Now, since we proved that $k \preceq a$. Hence by (8),

$$a \wedge (b \vee c) = ((a \to k) \vee ((b \vee c) \to k)) \to k = 1 \to k = k = (a \wedge b) \vee (a \wedge c).$$

Therefore, $(B(p), \wedge, \vee)$ is a distributive Lattice. ∎

Corollary 5.7. *If G_H is commutative, then for any $p \in G_H$, $(B(p), \vee, \wedge)$ is a Boolean lattice.*

Proof. By Theorems 5.4 and 5.6, the proof is clear. ∎

References

[1] Abbott, J.C., *Implicational algebras*, Bull. Math R.S. Roumaine, 11 (1967), 3-23.

[2] Abbott, J.C., *Sets, lattices and boolean algebras*, Allyn and Bacon, Boston, 1969.

[3] Birkhoff, G., *Lattice Theory*, American Mathematical Society, Colloquium Publications, XXV, 1940, third edition, 1967.

[4] Blyth, T.S., *Lattices and Orderd Algebraic Structurres*, Springer-Verlag, London, 2005.

[5] Celani, S., Cabrer, L., *Duality for finite Hilbert algebras*, Discrete Mathematics, 305 (2005), 74-99.

[6] Chen, C.C., Gratzer, G., *Stone lattices I. Construction theorems*, Canada Journal of Mathematics, 21 (1969), 884-894.

[7] Diego, A., *Sur les alg'ebres de Hilbert*, Collection de Logique Math'ematique, Serie A, 21, Gauthier-Villars, Paris, 1966.

[8] Halas, R., *Remarks on commutative Hilbert algebras*, Mathematica Bohemica, 4 (2002), 525-529.

[9] Henkin, L., *An algebraic characterization of quantifiers*, Fund. Math., vol. 37, (1950), 6374.

[10] Imai, Y., Iséki, K., *On axiom systems of propositional calculi*, XIV Proc. Japan Academy, 42 (1966), 19-22.

[11] Iséki, K., Tanaka, S., *An introduction to the theory of BCK-algebras*, Mathematicae Japonicae, 23 (1978), 1-26.

[12] Cirulis, J., *Hilbert algebras as implicative partial semilattices*, Central European Journal of Mathematics, 5(2) (2007), 264-279.

[13] Jun, Y.B., *Deductive systems of Hilbert algebras*, Scientiae Mathematicae Japonicae, 43 (1996), 51-54.

[14] Rasiwa, H., *An algebraic approach to non-classical logics*, Studies in logic and the foundations of mathematicas, 78, North-Holland, (1974).

Accepted: 07.11.2009