MULTI-OBJECTIVE DECISION MAKING BASED ON FUZZY EVENTS AND THEIR COHERENT (FUZZY) MEASURES

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Abstract. We propose a reformulation of the problem of making a decision with multiple objectives in terms of fuzzy scores and their consistent defuzzification, with respect to the logical point of view taken into account. The objectives are seen as subsets (or events) of a universal set U and the degree to which an alternative A_i satisfies the objective O_j is a conditional fuzzy event $A_i|O_j$, represented by a fuzzy set φ_{ij} defined on a partition π_{ij} of O_j. The elements of π_{ij} are the particular aspects of the objective O_j considered by A_i; the value assumed by an element $x \in \pi_{ij}$ is the extent to which A_i satisfies that particular aspect. Using an appropriate procedure of defuzzification fuzzy scores of alternatives with respect to the objectives are transformed into numerical scores belonging to the interval $[0, 1]$. We study the conditions of consistency of defuzzified scores taking into account the logical relations among the objectives and the alternatives. Finally, we develop criteria for the aggregation of scores of each alternative.

Keywords: multiobjective decision making, fuzzy events, coherent defuzzification, aggregation criteria.

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1. Introduction

A classical model of multi-objective decision making is based on a quadruple $(\mathbf{A}, \mathbf{O}, W, S)$, where $\mathbf{A}$ is the set of the *alternatives*, $\mathbf{O}$ is the set of the *objectives*, $W : \mathbf{O} \to [0, 1]$ is the *weight function*, which measures the weight, i.e. the importance, of the objectives; $S : \mathbf{A} \times \mathbf{O} \to [0, 1]$ is the *function score*, which, for each pair $(A_i, O_j) \in \mathbf{A} \times \mathbf{O}$, measures the score of A_i with respect to O_j, i.e. the extent to which the alternative A_i meets the objective O_j.

From now on we consider the case where $\mathbf{A}$ and $\mathbf{O}$ are finite. So we assume $\mathbf{A} = \{A_1, A_2, ..., A_m\}$, $\mathbf{O} = \{O_1, O_2, ..., O_n\}$, $W = \{w_1, w_2, ..., w_n\}$, where w_j is the weight of O_j.

Then the score function is represented by a matrix $S = (s_{ij})$, where s_{ij} measures the degree to which A_i meets the objective O_j. The rows of S are vectors associated with the alternatives and the columns are associated with the objectives.

Many authors, especially if they adopt the ranking procedure defined by the Analytic Hierarchy Process (AHP) [17], [3], [7], [11], [12], assume the conditions of normalization:

$$w_1 + w_2 + ... + w_n = 1, \tag{1}$$

$$\forall j \in 1, 2, ..., n, \ \ s_{1j} + s_{2j} + ... + s_{mj} = 1. \tag{2}$$

A classical formula to obtain the overall score $s(A_i)$ of the alternative A_i is as follows:

$$s(A_i) = w_1 s_{i1} + w_2 s_{i2} + ... + w_n s_{in}. \tag{3}$$

The preferred alternative is one that has the highest overall score.

We can observe that the role of objectives and alternatives is similar to that of events in subjective probability [5], [6], [4], [15]. Then let us extend the de Finetti's terminology to the decision problem. In particular, a family of objectives (resp. alternatives) two by two disjoint and exhaustive will be called *partition of the certain event.*

We note that, as in the subjective probability of de Finetti, each partition of the certain event is temporary, since each objective (resp. alternative) can be partitioned into sub-objectives (resp. sub-alternatives), i.e. for each partition of the certain event we can consider a finer. In this framework, an assignment of weights to a family of objectives (or alternatives) is similar to a probability assignment to a set of events, namely has the same formal properties, and then coherence conditions must be met. Then, from a formal point of view, the weight function $W : O_j \in \mathbf{O} \to w_j \in [0, 1]$ may be seen as a probability assignment on $\mathbf{O}$ and condition (1) follows from the assumption that $\mathbf{O}$ is a partition of the certain event Ω and W is a consistent assignment of probability [5], [4].

Similarly, the function $S : (A_i, O_j) \in \mathbf{A} \times \mathbf{O} \to s_{ij}$ plays the role of a conditional probability assignment [5], [4] where s_{ij} is the probability of the conditional event $A_i|O_j$. The conditions (2) follow from the hypothesis that $\mathbf{A}$ is a partition of the certain event and S is a coherent assessment of conditional probabilities. Then $s(A_i)$ can be interpreted (formally) as the probability of A_i and the formula (3) is a well-known formula of the theory of probability.

Then, in this order of ideas, if $\mathbf{O}$ (resp. $\mathbf{A}$) is not a partition of the certain event, consistency conditions are different from (1) (resp. (2)). They depend on the logical relationships between the events O_j (resp. A_i). These conditions reduce to the existence of nonnegative solutions of suitable linear systems [4].

In this paper we consider a more general point of view on the scores of the alternatives with respect to the objectives. The effect of an alternative A_i on an objective O_j is measured by a finite conditional fuzzy event $A_i|O_j$, represented by a fuzzy set φ_{ij} defined on a partition π_{ij} of O_j. The elements of π_{ij} are the particular aspects of the objective O_j considered by A_i; the value assumed by an element $x \in \pi_{ij}$ is the extent to which A_i satisfies that particular aspect.

The score s_{ij}, which provides an overall measure of the degree to which the alternative A_i meets the objective O_j, is interpreted as a defuzzification of φ_{ij}. In particular, in the notation of fuzzy events, s_{ij} is the probability of the conditional fuzzy event φ_{ij}.

The rest of the paper is organized as follows:

- In Section 2, we recall and introduce some concepts and results on fuzzy events and coherence conditions of their assignments of probability.
- In Section 3, we present a reformulation of the multi-objective decision making model in terms of fuzzy events and their coherent probabilities.
- In Section 4, we explore the problem of aggregation of the scores of each alternative with respect to the various objectives.
- In Section 5, we introduce fuzzy measures on fuzzy events, i.e., normalized monotonic measures, and we examine the conditions of consistency.
- Finally, in Section 6, we present some conclusions and research perspectives.

2. Fuzzy events and conditions of consistency of their assignments of probability

2.1. Basic concepts on fuzzy events

Fuzzy events and their assignments of probability were considered by Zadeh in [21] and [23]. The subjective probability of fuzzy events and conditions of coherence have been studied in [10]. In this subsection we introduce some basic concepts about fuzzy events, reworking and adapting the definitions given in [21] and [10], introducing some new concepts in view of the application to decision making problems.

Definition 2.1 A fuzzy event is a function $\varphi : \pi \to [0, 1]$, where π is a partition of the certain event Ω. For all $x \in \pi$, $\varphi(x)$ is the *degree* to which the fuzzy event φ occurred if x occurs.

Let $Im(\varphi)$ be the image of φ. For each $y \in Im(\varphi)$ we indicate with $\varphi^{(-1)}(y)$ the union of the elements x of π such that $\varphi(x) = y$. The set $\pi^* = \{\varphi^{(-1)}(y) : y \in Im(\varphi)\}$ is a partition of Ω and the function $\varphi^* : \pi^* \to [0, 1]$ such that $\varphi^*(x) = y$ if and only if $\varphi^{(-1)}(y) = x$ is a fuzzy event called the *reduced form* or *normal form* of φ. The elements $x \in \pi^*$ are called *atoms* or *constituents* of φ.

The fuzzy event φ is said to be *finite* if $Im(\varphi)$ is *finite*. In this case, if $\pi = \{x_1, x_2, ..., x_n\}, \varphi(x_i) = a_i$, using the notation of Zadeh [23] we write

$$\varphi = a_1/x_1 + a_2/x_2 + ... + a_n/x_n.$$

Remark 2.1 We note that, by identifying each event with its characteristic function, each event E can be seen as the fuzzy event φ_E with domain $\{E, E^c\}$ and such that $\varphi_E(E) = 1, \varphi_E(E^c) = 0$.

Definition 2.2 Let $\varphi_1 : \pi_1 \to [0, 1]$ and $\varphi_2 : \pi_2 \to [0, 1]$ two fuzzy events. We put:

$$\begin{gathered}\varphi_1 \leq \varphi_2 \Leftrightarrow \forall x_1 \in \pi_1, x_2 \in \pi_2, x_1 \cap x_2 \neq \emptyset \Rightarrow \varphi_1(x_1) \leq \varphi_2(x_2); \\ \varphi_1 = \varphi_2 \Leftrightarrow \varphi_1 \leq \varphi_2, \varphi_2 \leq \varphi_1.\end{gathered} \tag{4}$$

Remark 2.2 From the above definition it follows that two fuzzy events are equal if and only if they have the same reduced form. In particular, if two fuzzy events are equal then they have the same constituents.

If π_1 and π_2 are partitions of the certain event, let us denote with $\pi_1\pi_2$ their product, i.e. the partition $\{x_1 \cap x_2 : x_1 \in \pi_1, x_2 \in \pi_2, x_1 \cap x_2 \neq \emptyset\}$.

Definition 2.3 Let $\varphi_1 : \pi_1 \to [0, 1]$ and $\varphi_2 : \pi_2 \to [0, 1]$ be two fuzzy events and let $\star$ be an operation in $[0, 1]$. We define $\varphi_1 \star \varphi_2 : \pi_1\pi_2 \to [0, 1]$ as the fuzzy event with domain $\pi_1\pi_2$ and such that

$$\forall x_1 \in \pi_1, x_2 \in \pi_2 : x_1 \cap x_2 \neq \emptyset, (\varphi_1 \star \varphi_2)(x_1 \cap x_2) = \varphi_1(x_1) \star \varphi_2(x_2). \tag{5}$$

The most important are the following [8], [19]:

- $\star$ is a t-conorm, i.e. an operation in $[0, 1]$ associative, commutative, with 0 as neutral element and increasing in each variable;
- $\star$ is a t-norm, i.e. an operation in $[0, 1]$ associative, commutative, with 1 as neutral element and increasing in each variable.

Remark 2.3 We note that a constant $k \in [0, 1]$ is a fuzzy event φ with domain $\pi = \{\Omega\}$ and $\varphi(\Omega) = k$, and the multiplication in $[0, 1]$ is a t-norm. So the product of a fuzzy event by a scalar belonging to the interval $[0, 1]$ is a special case of the formula (5).

Definition 2.4 Let F be a nonempty family of fuzzy events such that

$$k \in [0, 1], \varphi \in F \Rightarrow k\varphi \in F,\ \varphi_1, \varphi_2 \in F, \varphi_1 + \varphi_2 \leq 1 \Rightarrow \varphi_1 + \varphi_2 \in F. \tag{6}$$

A function $p : F \to [0, 1]$ is said to be a *probability* on F if

P1 $\forall \varphi \in F, \inf(\varphi) \leq p(\varphi) \leq \sup(\varphi)$;

P2 $k \in [0, 1], \varphi \in F \Rightarrow p(k\varphi) = kp(\varphi)$;

P3 $\varphi_1, \varphi_2 \in F, \varphi_1 + \varphi_2 \leq 1 \Rightarrow p(\varphi_1 + \varphi_2) = p(\varphi_1) + p(\varphi_2)$.

As a consequence, we have the following corollary

Corollary 2.1 *Let $\varphi : \pi \to [0,1]$ be a finite fuzzy event, $\varphi = a_1/x_1 + a_2/x_2 + ... + a_n/x_n$. If $p : \pi \to [0,1]$ is a probability in π, then the probability of φ is the number:*

$$p(\varphi) = a_1 p(x_1) + a_2 p(x_2) + ... + a_n p(x_n). \tag{7}$$

2.2. Coherence of a probability assessment on fuzzy events

Definition 2.5 Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$ be a finite family of finite fuzzy events $\varphi_i : \pi_i \to [0,1]$ in reduced form. The elements of the product $\pi = \pi_1 \pi_2 ... \pi_h$ are called *atoms* or *constituents* of Φ.

Referring to the notations of the definition 2.5, let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of Φ. If $c_r = x_1^r \cap x_2^r \cap ... \cap x_h^r$, $x_i^r \in \pi_i$, we can write:

$$\varphi_i = a_i^1/c_1 + a_i^2/c_2 + ... + a_i^s/c_s, \ a_i^r = \varphi_i(x_i^r) = \varphi_i(c_r). \tag{8}$$

If the probabilities of the atoms were assigned then by the formula (7) we obtain the probability of each fuzzy event belonging to Φ.

In practical applications, however, often occurs that, based on information, beliefs, rationales, expert opinions, we have an assessment of the probabilities $p_i = p(\varphi_i)$ of the fuzzy events belonging to Φ without knowing the probabilities of the atoms. In this case the question arises of whether these judgments are *consistent*, i.e. if there exists a probability distribution on the atoms that permits to get the $p(\varphi_i)$ by the formula (7). For this purpose we give the following definition.

Definition 2.6 An assignment of probabilities $p = (p_1, p_2, ..., p_h)$ to the family of finite fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with $p_i = p(\varphi_i)$, is said to be coherent (or consistent) if there exists a probability distribution on the family of the atoms of Φ such that:

$$p(\varphi_i) = a_i^1 p(c_1) + a_i^2 p(c_2) + ... + a_i^s p(c_s). \tag{9}$$

with $a_i^r = \varphi_i(c_r)$.

Let $A = (a_{ir})$ be the matrix with $a_{ir} = \varphi_i(c_r)$ and let $P = [p_1, p_2, ..., p_h]^t$, $p_i = p(\varphi_i)$ be the column vector of probabilities assigned to the fuzzy events φ_i. Moreover let $Z = [z_1, z_2, ..., z_s]^t$ be the unknown vector of probabilities of the constituents. From the above definition the following theorem hold:

Theorem 2.1 *The assignment of probabilities $P = [p_1, p_2, ..., p_h]^t$ to the family of finite fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with $p_i = p(\varphi_i)$, is consistent if and only if there exists a solution of the system:*

$$AZ = P, \ z_1 + z_2 + ... + z_s = 1, \ Z \geq 0. \tag{10}$$

2.3. Conditional fuzzy events and coherence of their probability assessments

An extension of the concept of fuzzy event is conditional fuzzy event.

Definition 2.7 Let H be a non impossible event. A *fuzzy event conditional* on H is a function $\varphi : \pi \to [0,1]$, where π is a partition of the event H. If $\pi = \{x_1, x_2, ..., x_n\}$, $\varphi(x_i) = a_i$, using a notation consistent with that of Zadeh [23] we write:

$$\varphi = a_1/(x_1|H) + a_2/(x_2|H) + ... + a_n/(x_n|H). \tag{11}$$

For each $y \in Im(\varphi)$ we indicate with $\varphi^{(-1)}(y)$ the union of the elements x of π such that $\varphi(x) = y$. The set $\pi^* = \{\varphi^{(-1)}(y) : y \in Im(\varphi)\}$ is a partition of H and the function $\varphi^* : \pi^* \to [0,1]$ such that $\varphi^*(x) = y$ if and only if $\varphi^{(-1)}(y) = x$ is a conditional fuzzy event called the *reduced form* or *normal form* of φ. The non impossible elements $x \in \pi^* \cup \{H^c\}$ are called *atoms* or *constituents* of φ. The conditional events $x|H, x \in \pi^*$ are the *conditional atoms* (or *conditional constituents*) of φ.

For $H = \Omega$ previous definitions are reduced to that of (unconditional) fuzzy events. The definitions 2.2, 2.3, and their consequences extend to fuzzy events conditional on H by simply replacing Ω with H. Definition 2.4 extends to the case where F is a family of conditional fuzzy events with the same conditioning H and formula (7) is replaced by:

$$p(\varphi) = a_1 p(x_1|H) + a_2 p(x_2|H) + ... + a_n p(x_n|H) \tag{12}$$

where $p(x_i|H)$ is the probability of the conditional event $x_i|H$.

Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, $\varphi_i : \pi_i \to [0,1]$, be a finite family of finite fuzzy events conditional on H in reduced form and let $\pi = \pi_1\pi_2...\pi_h$. The non impossible events belonging to $\pi \cup \{H^c\}$ are called *atoms* or *constituents* of Φ and the conditional events $x|H, x \in \pi$ are the *conditional atoms.* If $\pi = \{c_1, c_2, ..., c_s\}$, $c_r = x_1^r \cap x_2^r \cap ... \cap x_h^r$, $x_i^r \in \pi_i$, we can extend formula (8) replacing the atoms c_r with the conditional atoms $c_r|H$:

$$\varphi_i = a_i^1/(c_1|H) + a_i^2/(c_2|H) + ... + a_i^s/(c_s|H),\ a_i^r = \varphi_i(x_i^r) = \varphi_i(c_r). \tag{13}$$

Formula (9) is replaced by:

$$p(\varphi_i) = a_i^1 p(c_1|H) + a_i^2 p(c_2|H) + ... + a_i^s p(c_s|H),\ a_i^r = \varphi_i(c_r). \tag{14}$$

Similarly, we can can extend Theorem 2.1. In this case, however, the z_i have the meaning of the unknown probabilities of the atoms conditional on H.

In order to connect the consistency of conditional fuzzy events with the coherence of (unconditional) fuzzy events, let us introduce the following definition.

Definition 2.8 Let $\varphi : \pi \rightarrow [0,1]$ a fuzzy event conditional on $H \neq \Omega$, $\varphi = a_1/(x_1|H) + a_2/(x_2|H) + ... + a_n/(x_n|H)$. We call (*unconditional*) *fuzzy event associated* with φ the fuzzy event $\varphi^0 : \pi^0 = \pi \cup \{H^c\} \rightarrow [0,1]$ defined as $\varphi^0 = a_1/x_1 + a_2/x_2 + ... + a_n/x_n + 0/H^c$.

Remark 2.4 It is well known that, for every pair of events (E, H), $E \subseteq H$, $H \neq \emptyset$, $p(E) = p(E|H)p(H)$. Then from (7) and (12) it follows that, if φ is a fuzzy event conditional on H and φ^0 is the (unconditional) fuzzy event associated, $p(\varphi^0) = p(\varphi)p(H)$.

Then, we have the following theorem:

Theorem 2.2 *Let $p = (p_1, p_2, ..., p_h)$ an assessment of probabilities to the family of finite conditional fuzzy events $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$, with φ_i conditional on H_i, $p_i = p(\varphi_i)$. If the assessment p implies that the probabilities of events H_i are all non-zero, then p is coherent if and only if the assessment $p^0 = (p_1 p(H_1), p_2 p(H_2), ..., p_h p(H_h))$ on the set of associated (unconditional) fuzzy events $\Phi^0 = (\varphi_1^0, \varphi_2^0, ..., \varphi_h^0)$ is coherent.*

Remark 2.5 It is worth noting that, from the theory on the consistency of conditional events [5], [4], [6] it follows that, if the assessment p does not imply that the probabilities of events H_i are all non-zero, then the consistency conditions on p are more complex than those of consistency of p^0.

3. A reformulation of the multi-objective decision model in terms of fuzzy events and their coherent probabilities

3.1. Weights and scores as coherent probabilities of fuzzy events

Assume, henceforth, that the objectives are events and that the effect of an alternative to an objective is represented by a finite conditional fuzzy event $A_i|O_j$ [10], [23], i.e. a fuzzy set $\varphi_{ij} : \pi_{ij} \rightarrow [0,1]$ with domain a finite partition π_{ij} of O_j. Each element $x \in \pi_{ij}$ is a particular aspect of the objective O_j and the value $\varphi_{ij}(x)$ is the extent to which the alternative A_i meets the facet x.

The score s_{ij}, which measures the degree to which, overall, the alternative A_i meets the objective O_j, is interpreted as a defuzzification of φ_{ij}. In particular, in this Sec., we assume that the scores s_{ij} meet, formally, the properties of a coherent assignment of probabilities on conditional fuzzy events φ_{ij}. In Sec. 5 we will consider other types of consistent defuzzification.

The weight w_j of the objective O_j is interpreted formally as its probability, so the product $w_j s_{ij}$ is the probability of the unconditional fuzzy event φ_{ij}^0 associated with φ_{ij}. If the weights w_j are positive, then by Theorem 2.2, the consistency of the assignment of probabilities s_{ij} on the conditional fuzzy events φ_{ij} is reduced to that of assigning coherent probabilities $w_j s_{ij}$ on the associated fuzzy events φ_{ij}^0.

Both logical reasons and to simplify the algorithms, you should first deal with the consistent assignment of weights $w_j > 0$ of the objectives O_j and then assigning consistent scores s_{ij} to the conditional fuzzy events φ_{ij}.

Let $\omega = \{o_1, o_2, ..., o_h\}$ be the set of atoms of the objectives O_j. Identifying the event O_j with its characteristic function we can write:

$$O_j = \delta_{j1}/o_1 + \delta_{j2}/o_2 + ... + \delta_{jh}/o_h,$$

with $\delta_{jr} = 1$ if $o_r \in O_j$, $\delta_{jr} = 0$ if $o_r \in O_j^c$.

Let $\Delta = (\delta_{jr})$ the matrix having as elements the numbers δ_{jr}. By Theorem 2.1, we have the following corollary:

Corollary 3.2 *The assignment of probabilities $W = [w_1, w_2, ..., w_n]^t$ to the events O_j is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_h]^t$ of the system of equations and inequalities:*

$$\Delta Z = W, \ z_1 + z_2 + ... + z_h = 1, \ Z \geq 0. \tag{15}$$

Let us remark that if the objectives are incompatible and exhaustive events then the consistency of weights reduces to condition (1).

Once assigned positive weights w_j to the objectives O_j in a coherent way we pass to the second part of the algorithm: consistently assign probabilities $w_j s_{ij}$ to fuzzy events φ_{ij}^0.

Let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of the fuzzy events φ_{ij}^0 and let $\varphi_{ij}^0(c_r) = a_{ij}^r$. By Theorem 2.1, we have:

Corollary 3.3 *The assignment of probabilities $w_j s_{ij}$ to fuzzy events φ_{ij}^0 is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_s]^t$ of the system of equations and inequalities:*

$$\forall (i,j), \ a_{ij}^1 z_1 + a_{ij}^2 z_2 + ... + a_{ij}^s z_s = w_j s_{ij} \tag{16}$$

$$z_1 + z_2 + ... + z_s = 1 \tag{17}$$

$$Z \geq 0 \tag{18}$$

If the scores are not consistent, then we must identify criteria and algorithms that allow us to gradually modify these scores and to get *closer to consistency* in every step.

There also seems useful to introduce the concept of *weak consistency*, which could replace the consistency in the case of complex decision problems with uncertain data.

3.2. Fuzzy coherence

Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_h\}$ be a finite family of finite fuzzy events $\varphi_i : \pi_i \to [0,1]$, $\pi = \{c_1, c_2, ..., c_s\}$ the set of the atoms. Let $A = (a_{ir})$ be the matrix with $a_{ir} = \varphi_i(c_r)$.

From Theorem 2.1, an assessment of probabilities $p = (p_1, p_2, ..., p_h)$ on Φ is coherent if and only if p belongs to the set

$$S = \{X = [x_1, x_2, ..., x_h]^t : X = AZ,\ Z \in [0,1]^s,\ z_1 + z_2 + ... + z_s = 1\}.$$

Let us call S the *set of coherence* (or *consistence*) associated with Φ. S is bounded, closed, and contained in $[0,1]^h$, then the Euclidean distance between p and S is a nonnegative real number $d(p,S)$ less than or equal to $\sqrt{h}$. The finding that distance reduces to a quadratic programming problem. In this framework let us give the following definition.

Definition 3.9 We define *fuzzy coherence* the fuzzy set

$$\gamma : p \in [0,1]^h \to 1 - \frac{d^2(p,S)}{h}.$$

For every $p \in [0,1]^h$, $\gamma(p)$ is the *degree of coherence* of p.

In practical applications, where there is uncertainty about the values of fuzzy events, it seems appropriate to accept a slight inconsistency. So, given a decision problem, we propose to set a suitable positive number $\alpha < 1$, depending on the complexity of the problem (e.g. $\alpha = 0.9$). An assignments of probabilities p to a family of fuzzy events is said to be *weakly consistent* if $\gamma(p) \geq \alpha$.

We believe that in complex decision problems with uncertain data can be permitted to accept weakly consistent assignments of probabilities.

3.3. An algorithm to get closer to the consistency

Let Φ be a finite family of h finite fuzzy events and S its set of coherence. We can find, for each $i \in \{1, 2, ..., h\}$, two points $P_i = (a_1, a_2, ..., a_h)$ and $Q_i = (b_1, b_2, ..., b_h)$ such that P_i is a solution of the linear programming problem:

$$\min x_i,\ (x_1, x_2, ..., x_h) \in S, \tag{19}$$

and Q_i is a solution of

$$\max x_i,\ (x_1, x_2, ..., x_h) \in S. \tag{20}$$

The interval $[a_i, b_i]$ is the projection of S on the axis x_i. Let T be the convex set generated by the points $P_i,\ Q_i,\ i \in \{1, 2, ..., h\}$. Let G be the barycenter of T.

If p is not a coherent probability assessment on Φ, we propose the following algorithm to get closer to the consistency:

(step 1) We fix a small positive real number ε, indicating the extent to which we approach the consistency in each iteration.

(step 2) We urge decision makers to update the assignment p with a new assignment q with the condition that the dot product between the vectors pq and pG is not less than ε (and thus, for the Euclidean distances, $d(q,G) \leq d(p,G) - \varepsilon$).

(step 3) We assign p = q. If p is consistent, the algorithm ends, if p is not consistent we return to step 2.

4. Aggregation of scores

A usual choice is to aggregate the scores of the alternatives using the formula (3). This is acceptable if the decision maker is aware that, in this way, the score of each constituent is counted as many times as there are objectives in which the constituent is contained. If the decision maker believes that this assumption is correct for the decision problem under discussion, then it is right to use the formula (3).

We remark that from formulae (3), (9) and (16) the global score of the alternative A_i is the number:

$$s(A_i) = \sum_{j=1}^{n} s_{ij} w_j = \sum_{r=1}^{s} [\sum_{j=1}^{n} a_{ij}^r] z_r. \tag{21}$$

where $a_{ij}^r = \varphi_{ij}^0(c_r)$.

This means that the score assigned to the atom c_r is:

$$s(c_r) = \sum_{j=1}^{n} a_{ij}^r, \tag{22}$$

i.e., it is the sum of the scores of c_r with respect to the alternative A_i in all the objectives containing c_r and

$$s(A_i) = \sum_{r=1}^{s} s(c_r) z_r. \tag{23}$$

There are many other criteria to assess the scores of atoms. For instance, if the decision maker wants the score of each constituent contained in at least an objective is counted only once in the aggregation of the scores of each alternative, he can assume that in formula (23) the score of the atom c_r with respect to the alternative A_i is:

$$s(c_r) = \max_{j=1}^{n} a_{ij}^r. \tag{24}$$

Of course, there are many other possible formulae for $s(c_r)$. Precisely, we can assume:

$$s(c_r) = f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r), \tag{25}$$

where f is a non negative real function, defined in $[0,1]^n$, null in $(0, 0, ..., 0)$, continuous, symmetric respect to every pair of variables, and increasing respect to every argument. For instance, the operation of "sum" or of "max" can be replaced by a t-conorm.

We emphasize that, if the formula (22) holds, then the value $s(A_i)$ in formula (23) is independent on the solution $(z_1, z_2, ..., z_s)$ considered of the system (16)–(17) with the conditions (18). On the contrary, if a different formula is adopted for $s(c_r)$, then the value $s(A_i)$ depend on $(z_1, z_2, ..., z_s)$. From the continuity of the function f, the set of values $s(A_i)$ is a closed interval $[m_i, M_i]$ of the real line.

Of course m_i is obtained when $(z_1, z_2, ..., z_s)$ is a solution P_i of the mathematical programming problem:

$$\min s(A_i) = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ z_r, \tag{26}$$

with the constraints given by system (16)–(17) with the conditions (18).

Similarly M_i is obtained when $(z_1, z_2, ..., z_s)$ is a solution Q_i of the mathematical programming problem:

$$\max s(A_i) = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ z_r, \tag{27}$$

with the constraints given by system (16) - (17) with the conditions (18).

We propose, below, to assume that $s(A_i)$ is a suitable triangular fuzzy number. For definitions and results on fuzzy numbers, see, e.g., [21], [22], [23], [8], [20].

It seems natural to assume the support of $s(A_i)$ is the closed interval $[m_i, M_i]$. In order to define the core $c(A_i)$ of the fuzzy number $s(A_i)$, we propose to consider the convex set $H = [P_i, Q_i, i \in \{1, 2, ..., m\}]$ generated by the vertices P_i, Q_i, $i \in \{1, 2, ..., m\}$. H is contained in the set K of all the solutions of the system (16)–(17) with the conditions (18), that is also a convex set.

Let $G = (g_1, g_2, ..., g_s)$ be the barycenter of H. G belongs to H and it seems reasonable to assume that the core of $s(A_i)$ is the value

$$h_i = \sum_{r=1}^{s} f(a_{i1}^r, a_{i2}^r, ..., a_{in}^r)\ g_r, \tag{28}$$

So we propose $s(A_i)$ is the triangular fuzzy number (m_i, h_i, M_i).

5. The multi-objective decision model in terms of fuzzy events and their fuzzy measures

5.1. Fuzzy measures

Let us recall the concepts of *fuzzy measure*, Archimedean t-conorms and decomposable measure and some basic results (see, e.g., [18], [19], [8], [9]).

Definition 5.10 Let U be a set and $\mathcal{E}$ a σ-field of subsets of U. A real function, $m : \mathcal{E} \to R$, is said to be a *fuzzy measure* on $\mathcal{E}$ if:

FM1 $m(\emptyset) = 0$; $m(U) = 1$;

FM2 $\forall A, B \in \mathcal{E}, A \subseteq B \Rightarrow m(A) \leq m(B)$;

FM3 if $\{A_n\}_{n \in N}$ is a monotonic sequence of elements of $\mathcal{E}$ then:

$$\lim_{n \to +\infty} A_n = A \Rightarrow \lim_{n \to +\infty} m(A_n) = m(A).$$

If $\mathcal{E}$ is finite then conditions FM1 and FM2 imply FM3. If we want to generalize the concept of finitely additive probabilities considered by de Finetti [5] then we must define a *weak fuzzy measure*, satisfying only the first two conditions, regardless of whether the domain is finite or not. Then we introduce the following definition.

Definition 5.11 Let U be a set and F a family of subsets of U containing $\{\emptyset, U\}$. A real function, $m : F \to R$, is said to be a *weak fuzzy measure* on F if:

FM1 $m(\emptyset) = 0$; $m(U) = 1$;

FM2 $\forall A, B \in F, A \subseteq B \Rightarrow m(A) \leq m(B)$.

Remark 5.6 A coherent finitely additive probability satisfies condition FM1 and FM2, then the concept of *weak fuzzy measure* is a generalization of *coherent finitely additive probability.*

Definition 5.12 A t-conorm $\oplus$ is said to be *Archimedean* if it is continuous and $x \oplus x > x$, $\forall x \in (0, 1)$. An Archimedean t-conorm is called *strict* if it is strictly increasing in the open square $(0, 1)^2$.

The following representation theorem holds [9]:

Theorem 5.3 *A binary operation $\oplus$ on $[0, 1]$ is an Archimedean t-conorm if and only if there exists a strictly increasing and continuous function $g : [0, 1] \to [0, +\infty]$, with $g(0) = 0$, such that*

$$x \oplus y = g^{(-1)}(g(x) + g(y)).$$

Function $g^{(-1)}$ denotes the pseudo-inverse of g, i.e.:

$$g^{(-1)}(x) = g^{-1}(\min(x, g(1))).$$

The function g, called an additive generator *of $\oplus$, is unique up to a positive constant factor. Moreover $\oplus$ is strict if and only if $g(1) = +\infty$.*

A compromise between the very general concept of weak fuzzy measure and that of finitely additive probability, rather restrictive in some applications of decision theory, was considered by some authors, notably by Weber [19]. Here are the definitions and basic results that will be useful for the rest of this paper.

Definition 5.13 Let U be a set and F a field of subsets of U. A weak fuzzy measure m on F is said to be *a measure decomposable* w. r. to a t-conorm $\oplus$, or *a $\oplus$-decomposable measure*, if:

$$A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) \oplus m(B).$$

In [19], the following classification theorem is proved:

Theorem 5.4 *If the operation $\oplus$ in $[0,1]$ is a strict Archimedean t-conorm, then $g \circ m : F \to [0,+\infty]$ is an infinite additive measure, whenever m is a $\oplus$-decomposable one.*

If $\oplus$ is a nonstrict Archimedean t-conorm, then $g \circ m$ is finite and one of the following cases occurs:

NSA *$g \circ m : F \to [0,+\infty]$ is a finite additive measure;*

NSP *$g \circ m$ is a finite set function which is only pseudo additive, i.e., if $\{A_n\}_{n \in \{1,2,\dots,s\}}$ is a family of pairwise disjoint elements of F, then:*

$$(g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) < g(1) \Rightarrow (g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) = \sum_{n=1}^{s}(g \circ m)(A_n);$$

$$(g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) = g(1) \Rightarrow (g \circ m)\left(\bigcup_{n=1}^{s} A_n\right) \leq \sum_{n=1}^{s}(g \circ m)(A_n).$$

5.2. An extension of fuzzy measures to fuzzy events

Let $\oplus$ be a nonstrict Archimedean t-conorm and let g be an additive generator of $\oplus$ with $g(1) = 1$. Let $\star$ be a t-norm. We introduce the following definition:

Definition 5.14 Let $\varphi : \pi \to [0,1]$ be a finite fuzzy event, $\varphi = a_1/x_1 + a_2/x_2 + \dots + a_n/x_n$. If $m : \pi \to [0,1]$ is a $\oplus$ - decomposable fuzzy measure in π, then the *measure of φ associated to the t-norm $\star$* is the number:

$$m(\varphi) = a_1 \star m(x_1) \oplus a_2 \star m(x_2) \oplus \dots \oplus a_n \star m(x_n). \tag{29}$$

Example 5.1 Two notable t-norms are the usual multiplication $\cdot$ and the t-norm $\cdot^g$, associated to the pair $(\cdot, g)$, defined as follows (see [8], p. 75)

$$a \cdot^g b = g^{(-1)}(g(a) \cdot g(b)). \tag{30}$$

5.3. Coherence of a fuzzy measure assessment on fuzzy events

If we replace probabilities with $\oplus$ - decomposable fuzzy measures, then Definition 2.6 is replaced by the following definition.

Definition 5.15 An assignment of $\oplus$-decomposable fuzzy measures $m = (m_1, m_2, ..., m_h)$ to the family of finite fuzzy events $\Phi = (\varphi_1, \varphi_2, ..., \varphi_h)$, with $m_i = m(\varphi_i)$, is said to be coherent (or consistent) if there exists a $\oplus$ - decomposable fuzzy measure distribution on the family $\{c_1, c_2, ..., c_s\}$ of the atoms of Φ such that:

$$m(\varphi_i) = a_i^1 \star m(c_1) \oplus a_i^2 \star m(c_2) \oplus ... \oplus a_i^s \star m(c_s), \tag{31}$$

with $a_i^r = \varphi_i(c_r)$.

Definition 5.15 and Theorem 5.4 imply the following theorem.

Theorem 5.5 *The assignment of $\oplus$-decomposable fuzzy measures $m = [m_1, m_2, ..., m_h]^t$, $m_i < 1$, to the family of finite fuzzy events $\Phi = (\varphi_1, \varphi_2, ..., \varphi_h)$, with $m_i = m(\varphi_i)$, is consistent if and only if there exists a solution of the system:*

$$\begin{aligned}
&(32) \quad \forall i \in \{1, 2, ..., h\}, g(a_i^1 \star z_1) + g(a_i^2 \star z_2) + ... + g(a_i^s \star z_s) &&= g(m_i) \\
&(33) \quad g(z_1) + g(z_2) + ... + g(z_s) &&\geq g(1) \\
&(34) \quad Z &&\geq 0
\end{aligned}$$

where z_r is the unknown measure of the atom c_r.

The previous system is not in general a linear system and is therefore difficult to solve. A substantial simplification is achieved, however, if the t-norm $\star$ is equal to $\cdot^g$. In fact, in this case it is reduced to the following system, linear with respect to the unknowns $g(z_r)$

$$\begin{aligned}
&(35) \quad \forall i, g(a_i^1)g(z_1) + g(a_i^2)g(z_2) + ... + g(a_i^s)g(z_s) &&= g(m_i) \\
&(36) \quad g(z_1) + g(z_2) + ... + g(z_s) = 1 &&\geq g(1) \\
&(37) \quad Z &&\geq 0.
\end{aligned}$$

5.4. Coherence of a fuzzy measure assessment as scores in a decision making problem

Let us refer to the notations used in Section 3.

Let $\pi = \{c_1, c_2, ..., c_s\}$ be the set of the atoms of the fuzzy events φ_{ij}^0 and let $\varphi_{ij}^0(c_r) = a_{ij}^r$.

From the results of the previous subsection, if the t-norm $\star$ is $\cdot^g$, then we have the following coherence theorem.

Theorem 5.6 *The assignment of $\oplus$-decomposable fuzzy measures $w_j s_{ij} < 1$, $w_j > 0$, to fuzzy events φ_{ij}^0 is consistent if and only if there exist solutions $Z = [z_1, z_2, ..., z_s]^t$ of the system of equations and inequalities:*

$$\begin{aligned}
&(38) \quad \forall (i,j),\ g(a_{ij}^1)g(z_1) + g(a_{ij}^2)g(z_2) + ... + g(a_{ij}^s)g(z_s) &&= g(w_j s_{ij}) \\
&(39) \quad g(z_1) + g(z_2) + ... + g(z_s) &&\geq g(1) \\
&(40) \quad Z &&\geq 0.
\end{aligned}$$

6. Conclusions and research perspectives

The aim of the paper is to stimulate a reflection on some key points in the decision process. In particular, we have explicated the hypotheses usually implicitly admitted in the decision-making processes and we have proposed criteria and procedures for assignment of weights and scores are consistent with the accepted assumptions and the logical relationships among the objectives and among the alternatives.

In the first 4 sections the reasoning and conclusions were bound by the idea of an additive aggregation of the weights or scores in a manner analogous to that which occurs in probability. In Sec. 5 were examined some implications arising from the idea of aggregations that follow logic other than additive.

The results can be helpful for the construction of consistent decision-making processes, i.e. taking into account the logical relations between objectives and alternatives, and the resulting numeric constraints in assigning weights and scores.

These constraints also depend on the ideas of measurement and aggregation of the measures that decision-makers see fit. We think it is important that these opinions and points of view are made explicit and that the assignments and criteria for aggregating measures adopted are consistent with these ideas.

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