DOMINATION IN THE INTERSECTION GRAPHS OF RINGS AND MODULES

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Abstract. In this paper we obtain domination number in the intersection graphs of ideals of rings, and intersection graphs of submodules of modules.

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1. Introduction

The graph theory and algebraic notation and terminology follow from [9] and [7], respectively. Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ of F is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$. It is known that every simple graph is an intersection graph, [8].

Let $G = (V, E)$ be a graph. The *(open) neighborhood* $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to v. For a subset S of vertices, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \sup S$. A set of vertices S in G is a *dominating set*, (or just DS), if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. For references on Domination Theory, see [6].

It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. Bosak [1], in 1964, studied graphs of semigroups. Then Csákány and Pollák [3], in 1969, studied the graphs of subgroups of a finite group. Zelinka [10], in 1975, continued the work on intersection graphs of nontrivial subgroups of finite abelian groups. Recently, Chakrabarty et al. [2] studied intersection graphs of ideals of rings. Jafari Rad et al. [4] considered intersection graph of subspaces of a vector space. They also studied the intersection graphs of submodules of a module [5].

In this paper, we study domination in the intersection graphs of ideals of rings and domination in the intersection graphs of submodules of modules. In section 2, we determine domination number in the intersection graphs of ideals of rings. In section 3, we determine the domination number in the intersection graphs of submodules of modules.

Throughout this paper, R is a commutative ring R with 1. For a ring R the intersection graph of ideals of R, denoted by $\Gamma(R)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial ideals of R and two distinct vertices are adjacent if and only if the corresponding ideals of R have a nontrivial (nonzero) intersection. For an R-module M, the intersection graph of submodules of M, denoted by $\Gamma(M)$, is the graph whose vertices are in a one-to-one correspondence with proper nontrivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules have a nontrivial (nonzero) intersection.

For a ring R, we define $\gamma(\Gamma(R)) = 0$ if R is a field, and for an R-module M, we define $\gamma(\Gamma(M)) = 0$ if M is simple.

2. Ring

In this section we determine the domination number in the intersection graphs of ideals of rings. We begin with the following obvious lemma.

Lemma 2.1 *Let R_1, R_2 be two rings with* 1. *Then $I \trianglelefteq R_1 \times R_2$ if and only if $I = I_1 \times I_2$, where $I_i \trianglelefteq R_i$ for $i = 1, 2$.*

Lemma 2.2 *Let R_1, R_2 be two rings with* 1. *Then $\gamma(\Gamma(R_1 \times R_2)) \leq 2$.*

Proof. Notice that $\{R_1 \times 0, 0 \times R_2\}$ is a DS of $\Gamma(R_1 \times R_2)$. ■

Theorem 2.3 *Let R_1, R_2 be two rings with* 1. *Then $\gamma(\Gamma(R_1 \times R_2)) = 1$ if and only if $\gamma(\Gamma(R_1)) = 1$ or $\gamma(\Gamma(R_2)) = 1$.*

Proof. ($\Longrightarrow$) Let $\{I \times J\}$ be a DS for $\Gamma(R_1 \times R_2)$. Since $R_1 \times 0$ and $0 \times R_2$ are dominated by $\{I \times J\}$, we obtain that I or J is a nontrivial proper ideal. Without loss of generality, assume that I is a nontrivial proper ideal. Now each ideal I_1 of R_1, $(I_1 \times 0) \cap (I \times J) \neq 0$. So $I \cap I_1 \neq 0$. This implies that $\{I\}$ is a DS for $\Gamma(R_1)$, and so $\gamma(\Gamma(R_1)) = 1$.

($\Longleftarrow$) Let $\gamma(\Gamma(R_1)) = 1$, and $\{I\}$ be a DS for $\Gamma(R_1)$. It follows that $\{I \times R_2\}$ is a DS for $\Gamma(R_1 \times R_2)$. This completes the proof. ■

A ring R is indecomposable if, for any pair of nontrivial rings R_1, R_2,

$$R \ncong R_1 \times R_2.$$

Lemma 2.4 *If R is an indecomposable ring with* 1 *such that $\Gamma(R) \neq \emptyset$, then $\gamma(\Gamma(R)) = 1$.*

Proof. Let M be a maximal ideal of R. Let I be an arbitrary proper nontrivial ideal of R. If $I \cap M = 0$ then $I + M = R$ and so $R \simeq I \times M$, a contradiction. So $I \cap M \neq 0$. We conclude that $\{M\}$ is a DS for $\Gamma(R)$. ∎

Corollary 2.5 *Let R be a ring with 1. Then $\gamma(\Gamma(R)) \leq 2$.*

Proof. The result follows from Lemmas 2.2 and 2.4. ∎

Theorem 2.6 *Let R be an Artinian commutative ring with 1. Then $\gamma(\Gamma(R)) = 2$ if and only if $R = R_1 \times R_2 \times ... \times R_t$, where $t \geq 2$ and R_i is a field for $i = 1, 2, ..., t$.*

Proof. ($\Longrightarrow$) Since R is Artinian, we have $R = R_1 \times R_2 \times ... \times R_t$, where R_i is a local ring for $i = 1, 2, ..., t$. By Theorem 2.3, $\gamma(\Gamma(R_i)) \neq 1$. This implies that $\Gamma(R_i)$ is the null graph, and so R_i is a field. But $\gamma(\Gamma(R)) = 2$. So $\Gamma(R) \neq \emptyset$, and then $t \geq 2$.

($\Longleftarrow$) Follows from Theorem 2.3. ∎

3. Module

In this section we determine the domination number in the intersection graphs of submodules of modules. An R-module M is *semisimple* if $M \cong M_1 \times M_2 \times ... \times M_k$, where M_i is a simple R-module for $i = 1, 2, ..., k$.

Lemma 3.7 *Let M be an Artinian R-module and $N =< \{K : K$ is a minimal submodule of $M\} >$. Then $N \cong K_1 \times K_2 \times ... \times K_t$, where K_i is minimal (simple).*

Proof. Assume to the contrary that $N \ncong K_1 \times K_2 \times ... \times K_t$, for any t and minimal submodules K_i. Let K_1 be a minimal submodule of M. Since $M \neq K_1$, there exists a minimal submodule K_2 such that $K_1 \cap K_2 = 0$. Since $N \neq K_1 \oplus K_2$, then there exists a minimal submodule K_3 such that $(K_1 + K_2) \cap K_3 = 0$. By continuing this method, we obtain minimal submodules $K_1, K_2, ...$ such that $\sum_{i \in \mathbb{N}} K_i = \bigoplus_{i \in \mathbb{N}} K_i$. Since $\bigoplus_{i \in \mathbb{N}} K_i$ is not Artinian R-module, we obtain a contradiction. Notice that K_i is simple, since it is minimal, for each i. ∎

Lemma 3.8 *If M is a semisimple module, then $\gamma(\Gamma(M)) \neq 1$.*

Proof. If M is simple, then $\gamma(\Gamma(M)) = 0$. So we assume that $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where M_i is simple and $t \geq 2$. Assume that $\{N\}$ is a DS for $\Gamma(M)$. For any i, $N \cap M_i \neq 0$, and so $M_i \subseteq N$. Then $M \subseteq N$, a contradiction. ∎

Corollary 3.9 *Let M be an Artinian R-module. Then $\gamma(\Gamma(M)) = 1$ if and only if M is not semisimple.*

Proof. The result follows from Lemmas 3.7 and 3.8. ∎

So in the rest of this section we consider semisimple modules.

Theorem 3.10 ([4]) *If V is a vector space of dimension $d \geq 2$ over a filed F of order q, then $\gamma(G(V)) = q + 1$.*

Lemma 3.11 *Let M be finite semisimple and $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where $k \geq 1$ and M_i is simple for $i = 1, 2, ..., k$. If for any i, j, $Ann(M_i) = Ann(M_j)$, then $\gamma(\Gamma(M)) = |M_1| + 1$.*

Proof. Let $m = Ann(M_1)$. Since M_1 is simple , m is maximal. By assumption, M is an $\frac{R}{m}$-module, and so is a vector space over $\frac{R}{m}$. By Theorem 3.10, $\gamma(\Gamma(M)) = \left|\frac{R}{m}\right| + 1 = |M_1| + 1$. ■

Lemma 3.12 *Let M be a semisimple R-module and $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where $k \geq 1$ and M_i is simple for $i = 1, 2, ..., k$. If $Ann(M_1) = Ann(M_2) = ... = Ann(M_t)$ and $Ann(M_1) \neq Ann(M_i)$ for $t + 1 \leq i \leq k$, where $t < k$, then any submodule N of M is in the form $N_1 + N_2$, where $N_1 \leq M_1 + M_2 + ...M_t$ and $N_2 \leq M_{t+1} + ... + M_k$.*

Proof. Let $x \in N$ and $x = x_1 + x_2$, where $x_1 \in M_1 + M_2 + ...M_t$ and $x_2 \in M_{t+1} + ... + M_k$. Let $Ann(M_i) = m_i$ for each i. Then $m_1 + (m_{t+1} \cap m_{t+2} \cap ... \cap m_k) = R$. This implies that there are $a \in m_1$ and $b \in m_{t+1} \cap m_{t+2} \cap ... \cap m_k$ such that $a + b = 1$. Now $bx = (1 - a)x_1 + bx_2 = x_1 - ax_1 + 0 = x_1$ and then $x_1 \in N$. This implies that $x_2 \in N$. This completes the proof. ■

Corollary 3.13 *Let M be a semisimple R-module and $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, where $k \geq 1$ and M_i is simple for $i = 1, 2, ..., k$. Assume that $Ann(M_1) = Ann(M_2) = ... = Ann(M_t)$ and $Ann(M_1) \neq Ann(M_i)$ for $t + 1 \leq i \leq k$, where $t < k$. Then $\gamma(\Gamma(M)) = 2$.*

Proof. By Corollary 3.9, $\gamma(\Gamma(M)) \geq 2$, and by Lemma 3.12, $\{M_1 + M_2 + ... + M_k, M_{k+1} + ... + M_t\}$ is a DS for $\Gamma(M)$. We conclude that $\gamma(\Gamma(M)) = 2$. ■

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