AN APPROACH TO THE APPROXIMATION OF THE INVERSE OF A SQUARE MATRIX BY HE'S HOMOTOPY PERTURBATION METHOD

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Abstract. In this paper, we present an efficient numerical algorithm for approximating the inverse of a square matrix based on homotopy perturbation method. Some numerical illustrations are given to show the efficiency of the algorithm.

1. Introduction

Approximating the inverse of square matrices specially the coefficient matrix of a system of linear equations has widespread applications in applied mathematics. In 1992, Liao [11] employed the basic ideas of homotopy to propose a general method for nonlinear problems, and modified it step by step [12]–[16].

This method has been successfully applied to solve many types of nonlinear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called "homotopy perturbation method" (HPM), is provided by He [7]–[10], as well as the recent developments[4]–[6].

In most cases, using HPM, gives a very rapid convergence of the solution series, and usually only a few iterations leading to very accurate solutions, specially when the modified one is applied [1], [17].

In this article, HPM is applied to the equation $\mathbf{AX} - \mathbf{I} = \mathbf{0}$ and the convergence of the method is considered under certain conditions.

2. Analysis of the method

Consider the equation

$$\mathbf{AX} - \mathbf{I} = \mathbf{0}, \qquad (1)$$

where

$$\mathbf{A} = [a_{ij}], \quad i = 1, 2, \cdots, n \quad , j = 1, 2, \cdots, n.$$

Let

$$\begin{aligned} L(\mathbf{U}) &= \mathbf{AU} - \mathbf{I}, \\ F(\mathbf{U}) &= \mathbf{U} - \mathbf{W_0}, \end{aligned}$$

we define homotopy $H(\mathbf{U}, p)$ by

$$\begin{aligned} H(\mathbf{U}, 0) &= F(\mathbf{U}), \\ H(\mathbf{U}, 1) &= L(\mathbf{U}), \end{aligned}$$

and typically, a convex homotopy as follows

$$H(\mathbf{U}, p) = (1 - p)F(\mathbf{U}) + pL(\mathbf{U}) = \mathbf{0}, \tag{2}$$

where F is an operator with known solution W_0. HPM uses the homotopy parameter p as an expanding parameter[4,6] to obtain

$$\mathbf{U} = \mathbf{U_0} + p\mathbf{U_1} + p^2\mathbf{U_2} + \cdots, \tag{3}$$

and for $p \to 1$, it gives an approximation to the solution of (1) as follows

$$\mathbf{V} = \lim_{p \to 1} (\mathbf{U_0} + p\mathbf{U_1} + p^2\mathbf{U_2} + \cdots).$$

By substituting (3) in (2) and equating the terms with identical powers of p, we obtain

$$\begin{aligned} p^0: \quad & \mathbf{U}_0 - \mathbf{W}_0 = \mathbf{0}, \\ & \mathbf{U}_0 = \mathbf{W}_0, \\ p^1: \quad & (\mathbf{A} - \mathbf{I})\mathbf{U}_0 + \mathbf{U}_1 - \mathbf{W}_0 - \mathbf{I} = \mathbf{0}, \\ & \mathbf{U}_1 = \mathbf{I} - (\mathbf{A} - \mathbf{I})\mathbf{U}_0 + \mathbf{W}_0, \\ p^2: \quad & (\mathbf{A} - \mathbf{I})\mathbf{U}_1 + \mathbf{U}_2 = \mathbf{0}, \\ & \mathbf{U}_2 = -(\mathbf{A} - \mathbf{I})\mathbf{U}_1 \end{aligned}$$

and in general

$$\mathbf{U}_{n+1} = -(\mathbf{A} - \mathbf{I})\mathbf{U}_n \quad , \quad n = 1, 2, \cdots,$$

if we take $\mathbf{U}_0 = \mathbf{W}_0 = 0$, then we have

$$\begin{aligned} \mathbf{U_1} &= \mathbf{I}, \\ \mathbf{U}_2 &= -(\mathbf{A} - \mathbf{I})\mathbf{U}_1, \\ &= (\mathbf{A} - \mathbf{I}), \\ \mathbf{U}_3 &= (\mathbf{A} - \mathbf{I})^2, \\ &\vdots \\ \mathbf{U}_{n+1} &= (-1)^n(\mathbf{A} - \mathbf{I})^n, \end{aligned}$$

hence, the solution can be of the form

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \cdots ,$$

or

$$\mathbf{U} = [\mathbf{I} - (\mathbf{A} - \mathbf{I}) + (\mathbf{A} - \mathbf{I})^2 - \cdots]. \tag{4}$$

Theorem 1. *The sequence*

$$\mathbf{U}^{[m]} = \left[\sum_{k=0}^{m} ((\mathbf{A} - \mathbf{I})^k \right],$$

is a Chauchy sequence if

$$\| \mathbf{A} - \mathbf{I} \| < 1.$$

Proof. We must show that

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| = 0,$$

so, for showing this we can write

$$\mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} = \sum_{k=1}^{p} (-1)^{m+k} (\mathbf{A} - \mathbf{I})^{m+k},$$

or

$$\| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| \leq \sum_{k=1}^{p} \| \mathbf{A} - \mathbf{I} \|^{m+k},$$

let $\gamma = \| \mathbf{A} - \mathbf{I} \|$, then

$$\begin{aligned} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| &\leq \gamma^m \sum_{k=1}^{p} \gamma^k, \\ &\leq \left(\frac{\gamma^p - 1}{\gamma - 1} \right) \gamma^m, \end{aligned}$$

now if $\gamma < 1$, then we have

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| \leq \left(\frac{\gamma^p - 1}{\gamma - 1} \right) \left(\lim_{m\to\infty} \gamma^m \right),$$

hence, we obtain

$$\lim_{m\to\infty} \| \mathbf{U}^{[m+p]} - \mathbf{U}^{[m]} \| = 0,$$

which completes the proof. ∎

Lemma 1. *If* $\mathbf{A}$ *is diagonally dominated and*

$$\mathbf{D} = diag\left[\frac{1}{a_{ii}}\right],$$

then

$$\| \mathbf{DA} - \mathbf{I} \|_\infty < 1.$$

Proof. Let $\mathbf{C} = \mathbf{DA} - \mathbf{I}$, then it can be easily shown that

$$c_{ij} = \begin{cases} 0 & i = j \\ \dfrac{a_{ij}}{a_{ii}} & i \neq j \end{cases},$$

since $\mathbf{A}$ is diagonally dominated, then

$$| a_{ii} | > \sum_{j=1, j\neq i}^{n} | a_{ij} |,$$

or

$$\sum_{j=1, j\neq i}^{n} \frac{| a_{ij} |}{| a_{ii} |} < 1 \quad , \quad i = 1, 2, \cdots n,$$

hence

$$\sum_{j=1}^{n} | c_{ij} | < 1 \quad , \quad i = 1, 2 \cdots n,$$

which implies

$$\| \mathbf{C} \|_\infty = \| \mathbf{DA} - \mathbf{I} \|_\infty < 1.$$

If $\| \mathbf{A} - \mathbf{I} \|_\infty > 1$, by pre-multiplying the both sides of equation(1) by matrix $\mathbf{D}$, we rewrite the equation as follows

$$\mathbf{DAX} - \mathbf{D} = \mathbf{0}$$

and it can be easily shown that

$$\mathbf{U}_{n+1} = (-1)^n (\mathbf{DA} - \mathbf{I})^n \mathbf{D} \tag{5}$$

3. Numerical results

Example 1. Approximate the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 1 \\ 0 & 1 & -3 \end{bmatrix}.$$

Since $\| \mathbf{A} - \mathbf{I} \|_\infty = 8$, and $\mathbf{A}$ is diagonally dominated, we can write

$$D = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix},$$

hence

$$\mathbf{DA} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 1 & \frac{1}{6} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$\mathbf{DA} - \mathbf{I} = \begin{bmatrix} 0 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 0 & \frac{1}{6} \\ 0 & -\frac{1}{3} & 0 \end{bmatrix},$$

from (4)and (5) we have

$$\mathbf{U} = [\mathbf{I} - (\mathbf{DA} - \mathbf{I}) + (\mathbf{DA} - \mathbf{I})^2 - \cdots]\mathbf{D},$$

and using six terms, we approximate $\mathbf{A^{-1}} = \mathbf{U}$ as follows

$$\mathbf{U} \approx \mathbf{U}_0 + \mathbf{U}_1 + \cdots + \mathbf{U}_5,$$

or

$$\mathbf{U} \approx \begin{bmatrix} 0.2435 & -0.025 & -0.089 \\ 0.0385 & 0.1539 & 0.0387 \\ 0.0127 & 0.0514 & -0.32 \end{bmatrix},$$

where the exact solution is

$$\mathbf{A^{-1}} = \begin{bmatrix} 0.2435 & -0.025 & -0.089 \\ 0.0384 & 0.1538 & 0.0384 \\ 0.0128 & 0.0512 & -0.32 \end{bmatrix}$$

Example 2. Give an approximation to the inverse of the following matrix

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Matrix $\mathbf{A}$ is nearly diagonally dominated, so by similar operations we obtain

$$\mathbf{DA} - \mathbf{I} = \begin{bmatrix} 0 & 1 & 0.4 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix},$$

and using seven iterations, an approximation to $\mathbf{A^{-1}}$ will be as follows

$$\begin{bmatrix} 3.1956 & -4.4739 & -.15056 \\ -0.66063 & 5.19744 & -0.97097 \\ -0.66063 & 0.19973 & 4.02674 \end{bmatrix},$$

where the exact solution is

$$\mathbf{A^{-1}} = \begin{bmatrix} 3.07692 & -5 & -0.38462 \\ -0.76923 & 5 & -1.15385 \\ -0.76923 & 0 & 3.84615 \end{bmatrix}.$$

4. Conclusion

In this paper, we used homotopy perturbation method to approximate the inverse of a diagonally dominated matrix in terms of the subtraction of the given matrix (or its modification) and unit matrix. Solved problems show the convergence of the method increases as the the matrix becomes more strictly diagonally dominated.

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