ON A CLASS OF CHINESE HYPERRINGS

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Abstract. In this paper we have shown one construction of the Chinese hyperrings, using the class of multiendomorphisms of the starting ring $(R, +, \cdot)$ or the class of multiendomorphisms of its additive group $(R, +)$.

Key words: hyperring, hypergroup, multiendomorphism.

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1. Introduction

This paper deals with a class of Chinese hyperrings. Some Chinese mathematicians have developed an interesting theory of HX-groups. P. Corsini [4] showed that from every HX-group, a hypergrupoid is obtained which is, in some cases, a hypergroup (Chinese hypergroup). Similarly, we show that, if we start from the ring $(R, +, \circ)$, then each family $\mathcal{R} \subset P_0(R)$, that satisfies conditions of Theorem 3.1, generates a Chinese hyperring $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$. Theorem 3.5 shows that each multiendomorphism of the additive group $(R, +)$, under certain conditions, generates the family $\mathcal{R} \subset P_0(R)$ that satisfies the conditions of Theorem 3.1. So, there exists a class of multiendomorphisms of the additive group $(R, +)$, that generates a class of Chinese hyperrings. Theorem 3.10 shows that between the starting ring $(R, +, \circ)$ and the appropriate Chinese hyperring $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$, we can establish a good multihomomorphism.

2. Preliminaries

Definition 2.1. A hypergroupoid $(H, \circ)$ is a non-empty set H equipped with a function $o : H \times H \to P^*(H)$ called a hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H. If A, B are subsets of H, then $A \circ B$ denotes the set $\bigcup\limits_{\substack{a\in A\\ b\in B}} a \circ b$.

Definition 2.2.

1) An H_v-semigroup is a hypergroupoid $(H, \circ)$ such that:

$$\forall (x, y, z) \in H^3, \quad (x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset.$$

2) An H_v-semigroup is called an H_v-group if

$$\forall a \in H, \quad H \circ a = a \circ H = H.$$

3) A hyperstructure $(H, +, \circ)$ is called an H_v ring if:

   i) $(H, +)$ is an H_v-group.
   ii) $(H, \circ)$ is an H_v-semigroup.
   iii) For all x, y, z in H we have:

$$x \circ (y + z) \cap (x \circ y + x \circ z) \neq \emptyset \text{ and } \quad (y + z) \circ x \cap (y \circ x + z \circ x) \neq \emptyset.$$

Definition 2.3. A hypergroupoid $\langle H; o \rangle$ is called a semihypergroup if:

$$\forall (x, y, z) \in H^3 : (x \circ y) \circ z = x \circ (y \circ z).$$

Definition 2.4. A semi-hypergroup $\langle H; o \rangle$ is called a hypergroup (or also a multigroup) if

$$(\forall a \in H) \;\; H \circ a = a \circ H = H.$$

The previous axiom is called the *reproduction axiom*.

Definition 2.5. A multivalued system $(R, +, \circ)$ is a hyperring if:

1) $(R, +)$ is a hypergroup.

2) $(R, \circ)$ is a semihypergroup.

3) For all x, y, z in R, we have:

   (a) $x \circ (y + z) \subseteq x \circ y + x \circ z$
   and
   (b) $(y + z) \circ x \subseteq y \circ x + z \circ x$.

If in the conditions 3a) and 3b) the equality is valid, the hyperring is called *strongly distributive*.

Definition 2.6. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. The map $f : A \to B$ is called an *inclusion homomorphism* if for all $x, y \in A$, the following relations hold:

$$f(x + y) \subseteq f(x) +' f(y) \text{ and } f(x \cdot y) \subseteq f(x) \cdot' f(y).$$

The map f is called a *good homomorphism* if for all $x, y \in A$, we have:

$$f(x+y) = f(x) +' f(y) \quad \text{and} \quad f(x \cdot y) = f(x) \cdot' f(y).$$

Definition 2.7. Let $(A, +, \cdot)$ and $(B, +', \cdot')$ be two hyperrings. The map $\Phi : A \to P^*(B)$ is called a *multihomomorphism from the hyperring A to the hyperring B* if, for all $x, y \in A$, the following relations hold:

1) $\displaystyle\bigcup_{u \in x+y} \Phi(u) \subseteq \Phi(x) +' \Phi(y)$

2) $\displaystyle\bigcup_{u \in x \cdot y} \Phi(u) \subseteq \Phi(x) \cdot' \Phi(y).$

If in the conditions 1) and 2) the equality is valid, then the map Φ is called a *good multihomomorphism*.

Let us notice that, if $(A, +, \cdot)$ and $(B, +', \cdot')$ are the rings, then the conditions 1) and 2) means:

$$(\forall\, x, y \in A) \;\; \Phi(x+y) = \Phi(x) +' \Phi(y) \;\; \text{and} \;\; \Phi(xy) = \Phi(x) \cdot' \Phi(y).$$

Let $(G, \cdot)$ be an arbitrary group and $P(G)$ the powerset of G. Under the subset multiplication:

$$(\forall\, A, B \in P_0(G))) \; A \cdot B = \{ab \mid a \in A, \; b \in B\}.$$

$P_0(G) = P(G) \setminus \{\emptyset\}$ forms a semigroup which has an identity.

Definition 2.8. [2] A subgroup $(\mathcal{G}, \cdot)$ of $P_0(G)$ is called an *HX-group* on G and $(G, \cdot)$ the *generating group* (or the *support*) of $(\mathcal{G}, \cdot)$. The identity in $(\mathcal{G}, \cdot)$ is denoted by E.

Definition 2.9. [2] Let $(\mathcal{G}, \cdot)$ be an HX-group on $(G, \cdot)$. $(\mathcal{G}, \cdot)$ is called a *regular HX-group* if $e \in E$ (e is the identity of G).

Let $A \in \mathcal{G}$ and A^{-1} be the inverse element of A. The set $A^{(-1)} = \{x^{-1} \mid x \in A\}$ is called the *inverse set* of A.

Definition 2.10. [2] An HX-group $(\mathcal{G}, \cdot)$ is called a *uniform HX-group* if for all $A \in \mathcal{G}$ it holds:

$$A^{-1} = A^{(-1)}.$$

Theorem 2.11. [2] *An HX-group $(\mathcal{G}, \cdot)$ is a uniform HX-group iff E is a subgroup of G.*

Definition 2.12. [4] Let $(\mathcal{G},\cdot)$ be an HX-group with $(G,\cdot)$ as support and E as identity. We call a *Chinese hypergroupoid* the hyperstructure $\langle G^*,\widehat{\circ}\rangle$, where $G^*=\bigcup\limits_{A\in\mathcal{G}} A$ and

$$\forall\,(x,y)\in G^*\times G^*,\quad x\widehat{\circ}y=\bigcup_{\substack{x\in A,\ y\in B\\ \{A,B\}\subset\mathcal{G}}} A\cdot B.$$

Let $(\forall\,x\in G^*)$, $\alpha(x)=\{A\mid A\in\mathcal{G},\ A\ni x\}$ and $A(x)=\bigcup\limits_{A\in\alpha(x)} A.$

Lema 2.13. [4] $\forall\,(x,y)\in G^*\times G^*$, *we have:*

$$x\widehat{\circ}y=A(x)\cdot A(y).$$

Theorem 2.14. [4] *The hypergroupoid $\langle G^*,\widehat{\circ}\rangle$ is an H_v-group.*

Theorem 2.15 [4] *If $(\mathcal{G},\cdot)$ is an HX-group such that*

$$\forall\,(A,B)\in\mathcal{G}\times\mathcal{G},\quad A\cap B\neq\emptyset\Longrightarrow A=B,$$

then $\langle G^*,\widehat{\circ}\rangle$ is a hypergroup.

3. Chinese hyperrings

Theorem 3.1.

1) Let $(R,+,\cdot)$ be a ring and $\mathcal{R}\subset P_0(R)$ additive HX-group i.e. $\mathcal{R}$ is a group with respect to the operation:

$$(\forall\,A,B\in\mathcal{R})\ \ A+B=\{a+b\mid a\in A,\ b\in B\}.$$

Let $(\mathcal{R},\cdot)$ be a semigroup with respect to the operation:

$$(\forall\,A,B\in\mathcal{R})\ \ A\cdot B=\{ab\mid a\in A,\ b\in B\}.$$

On the set $\mathcal{R}^*=\bigcup\limits_{A\in\mathcal{R}} A$, we define the operations $\widehat{\oplus}$ and $\widehat{\odot}$:

$$(\forall\,x,y\in\mathcal{R}^*)\ \ x\widehat{\oplus}y=A(x)+A(y)\ \text{ and }\ x\widehat{\odot}y=A(x)\cdot A(y),$$

where

$$A(x)=\bigcup_{\substack{A\ni x\\ A\in\mathcal{R}}} A.$$

The structure $(\mathcal{R}^*,\widehat{\oplus},\widehat{\odot})$ is an H_v-ring.

2) If $\mathcal{R} \subset P_0(R)$ beside previous assumptions satisfies the condition:

$$(\forall\, A, B \in \mathcal{R}) \quad A \cap B \neq \emptyset \Longrightarrow A = B \tag{$**$}$$

then the structure $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring. (This hyperring we will call the Chinese hyperring).

Proof. Theorem 2.15 shows that $(\mathcal{R}^*, \widehat{\oplus})$ is an H_v-group.

We can show that $(\mathcal{R}^*, \widehat{\odot})$ is an H_v-semigroup using the method of P. Corsini [4]. Indeed, if $z \in x\widehat{\odot}y = A(x) \cdot A(y) = \left(\bigcup\limits_{\substack{x\in A\\ A\in\mathcal{R}}} A\right) \cdot \left(\bigcup\limits_{\substack{y\in B\\ B\in\mathcal{R}}} B\right)$, then A, B exist in $\mathcal{R}$ such that $x \in A$, $y \in B$ and $z \in A \cdot B$. Since $(\mathcal{R}, \cdot)$ is a semigroup, it is clear that $A \cdot B \in \mathcal{R}$, i.e., $z \in A \cdot B \subseteq \mathcal{R}^*$.

Let $x, y, z \in \mathcal{R}^*$. Then, we have:

$$\begin{aligned}(x\widehat{\odot}y)\widehat{\odot}z &= (A(x) \cdot A(y))\widehat{\odot}z = \bigcup_{u\in A(x)\cdot A(y)} u\widehat{\odot}z \\ &= \bigcup_{u\in A(x)\cdot A(y)} A(u) \cdot A(z) = \bigcup_{\substack{u\in A(x)\cdot A(y)\\ z\in A_3}} A(u) \cdot A_3 \\ &= \bigcup_{\substack{A_u \ni u \in \dot{A}_1 \dot{A}_2 \\ x\in\dot{A}_1 \cdot y\in\dot{A}_2 \cdot z\in A_3}} A_u \cdot A_3 = (*).\end{aligned}$$

So, we proved: $\forall\,(A_1, A_2, A_3) \in \mathcal{R}^3$ such that $x \in A_1$, $y \in A_2$, $z \in A_3$, it holds:

$$(*) \supseteq (A_1A_2)A_3 \;\text{ i.e. }\; (x\widehat{\odot}y)\widehat{\odot}z \supseteq (A_1A_2)A_3.$$

Similarly, we can prove: $\forall\,(A_1, A_2, A_3) \in \mathcal{R}^3$ such that $x \in A_1$, $y \in A_2$, $z \in A_3$, it holds:

$$x\widehat{\odot}(y\widehat{\odot}z) \supseteq A_1(A_2A_3).$$

Therefore, $(x\widehat{\odot}y)\widehat{\odot}z \cap x\widehat{\odot}(y\widehat{\odot}z) \neq \emptyset$, i.e. $(\mathcal{R}^*, \widehat{\odot})$ is an H_v-semigroup.

Let us prove now that $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$ satisfies the condition

$$(x\widehat{\oplus}y)\widehat{\odot}z \cap ((x\widehat{\odot}z)\widehat{\oplus}(y\widehat{\odot}z)) \neq \emptyset$$

for all $x, y, z \in \mathcal{R}^*$.

Let $x, y, z \in \mathcal{R}^*$. Then we have:

$$L = (x\widehat{\oplus}y)\widehat{\odot}z = (A(x)+A(y))\widehat{\odot}z = \bigcup_{u\in A(x)+A(y)} u\widehat{\odot}z = \bigcup_{\substack{A_u \ni u\in\dot{A}_1+\dot{A}_2\\ x\in\dot{A}_1,\ y\in\dot{A}_2,\ z\in A_3}} A_u \cdot A_3 = (*)'.$$

We can notice that for arbitrary $(A_1, A_2, A_3) \in \mathcal{R}^3$, where $x \in A_1$, $y \in A_2$ and $z \in A_3$, it holds:

$$(*)' \supseteq (A_1 + A_2) \cdot A_3, \;\text{ i.e. }\; L \supseteq (A_1 + A_2) \cdot A_3. \tag{1}$$

Also we have:

$$D = (x\widehat{\odot}z)\widehat{\oplus}(y\widehat{\odot}z) = (A(x)\cdot A(z))\widehat{\oplus}(A(y)\cdot A(z)) = \bigcup u\widehat{\oplus}v,$$

where,

$$u \in A(x)\cdot A(z) = \bigcup_{\substack{x\in A_1\\ z\in A_3}} A_1\cdot A_3 \quad\text{and}\quad v\in A(y)\cdot A(z) = \bigcup_{\substack{y\in A_2\\ z\in a_3}} A_2\cdot A_3.$$

Thus,

$$D = \bigcup_{\substack{A_u\ni u\in A_1\cdot A_3,\ x\in A_1,\ z\in A_3\\ A_v\ni v\in A_2\cdot A_3,\ y\in A_2,\ z\in A_3}} A_u + A_v.$$

Therefore, for arbitrary $(A_1, A_2, A_3)\in\mathcal{R}^3$, where $x\in A_1$, $y\in A_2$ and $z\in A_3$, it holds:

$$D \supseteq A_1\cdot A_3 + A_2\cdot A_3 \supseteq (A_1 + A_2)\cdot A_3. \tag{2}$$

From (1) and (2), it follows that $L\cap D\neq\emptyset$.

Similarly, we can show that for arbitrary $x, y, z\in\mathcal{R}^*$ it holds:

$$x\widehat{\odot}(y\widehat{\oplus}z)\cap((x\widehat{\odot}y)\widehat{\oplus}(x\widehat{\odot}z))\neq\emptyset.$$

So, we proved that $(\mathcal{R}^*, \widehat{\oplus}, \widehat{\odot})$ is an H_v-hyperring.

2) It is enough to remark that the condition $(**)$ implies $(\forall\, x\in\mathcal{R}^*)\ |\alpha(x)| = 1$, where

$$\alpha(x) = \{A \mid A\in\mathcal{R},\ A\ni x\}.$$

Let $(G,\cdot)$ and $(G_1,\circ)$ be two arbitrary groups, and let e be an identity element of $(G,\cdot)$.

Definition 3.2. [3] A multivalued mapping $f: G\to P_0(G_1) = \{B\subset G_1 \mid B\neq\emptyset\}$ such that:

$$(\forall\, x, y\in G)\quad f(x\cdot y) = f(x)\circ f(y) = \{a\circ b \mid a\in f(x), b\in f(y)\}$$

is called a *multihomorphism* of group G into a group G_1.

It is easy to see, that for arbitrary $x, y\in G$ it holds:

1) $f(x)\circ f(y) = f(xy)$

2) $(f(x)\circ f(y))\circ f(z) = f(xyz) = f(x)\circ(f(y)\circ f(z))$

3) $f(x) = f(x)\circ f(e) = f(e)\circ f(x)$

4) $f(e) = f(x)\circ f(x^{-1}) = f(x^{-1})\circ f(x)$.

Theorem 3.3. [3] *Let f be a multihomomorphism of the group $(G,\cdot)$ into the group $(G_1,\circ)$, such that $f(e)$ is subgroup of G_1. Then it holds:*

$$(\forall\, x,y\in G)\quad f(x)\cap f(y)\neq\emptyset \Longrightarrow f(x)=f(y).$$

We prove this theorem using the theory of HX-groups.

Proof. Let $\mathcal{G}_1=\{f(x)\mid x\in G\}$. Then it is easy to see that $\mathcal{G}_1$ is an HX-group on G_1, with identity $E=f(e)$, (e is the identity of G). Since, $E=f(e)$ is subgroup of G_1, then from Theorem 2.11 it follows that $\mathcal{G}_1$ is a uniform HX-group, i.e, it holds:

$$(f(x))^{(-1)}=(f(x))^{-1}=f(x^{-1}). \tag{$*$}$$

Now, suppose that $z\in f(x)\cap f(y)$.

From $z\in f(y)$, it follows that $z^{-1}\in (f(y))^{(-1)}\overset{(*)}{=}f(y^{-1})$.

Let e_1 be the identity of $(G_1,\circ)$. From

$$e_1=z\circ z^{-1}\in f(x)\circ f(y^{-1})=f(xy^{-1}),$$

it follows $f(e)=e_1\circ f(e)\subseteq f(xy^{-1})\circ f(e)=f(xy^{-1}e)=f(xy^{-1})$. Therefore,

$$f(e)\subseteq f(xy^{-1}) \tag{1}$$

Similarly,

$$f(e)\subseteq f(yx^{-1}). \tag{2}$$

From (1) and (2) it follows that:

$$f(xy^{-1})\supseteq f(e)=f(xy^{-1}yx^{-1})=f(xy^{-1})\circ f(yx^{-1})\supseteq f(xy^{-1})\circ f(e)=f(xy^{-1}),$$

i.e. $f(xy^{-1})\supseteq f(e)\supseteq f(xy^{-1})$. Thus, $f(xy^{-1})=f(e)$. Therefore,

$$f(x)=f(xy^{-1}y)=f(xy^{-1})f(y)=f(e)f(y)=f(y).$$

Corollary 3.4. *Let f be a multihomomorphism of the group $(G,\cdot)$ into the group $(G_1,\circ)$ such that $f(e)$ is subgroup of G_1. Then, for arbitrary $x\in G$ it holds:*

$$f(x)=a\circ f(e)=f(e)\circ a,$$

where a is an arbitrary element of $f(x)$.

Proof. Let $a\in f(x)$. Then:

$$a\circ f(e)\subseteq f(x)\circ f(e)=f(xe)=f(x)$$

and

$$f(e)\circ a\subseteq f(e)\circ f(x)=f(ex)=f(x).$$

Conversely, if $b \in f(x)$, then it holds:

$$b = a \circ (a^{-1} \circ b).$$

Let us prove that $a^{-1} \circ b \in f(e)$. From $a \in f(x)$ it follows that $a^{-1} \in (f(x))^{(-1)} = f(x^{-1})$ and thus $a^{-1} \circ b \in f(x^{-1}) \circ f(x) = f(e)$ and for this reason $b \in a \circ f(e)$.

We proved that $f(x) \subseteq a \circ f(e)$. Therefore, $a \circ f(e) = f(x)$. Similarly, we can prove that $f(e) \circ a = f(x)$.

Theorem 3.5. *Let $(R, +, \cdot)$ be a ring and $f : R \to P_0(R)$ a multiendomorphism of the additive group $(R, +)$ such that $f(0)$ is a subgroup of a group $(R, +)$. Let $\mathcal{R}_f = \{f(x) \mid x \in R\}$ and $\mathcal{R}_f^* = \bigcup_{x \in R} f(x)$. If the following condition is valid:*

$$(\forall\, x, y \in R) \;\; (\exists z \in R) \;\; f(x) \cdot f(y) = f(z) \tag{1}$$

and if we define on $\mathcal{R}_f^*$ the hyperoperations $\widehat{\oplus}$ and $\widehat{\odot}$ as in Theorem 3.1, then the structure $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.

Proof. Let us notice that $(\mathcal{R}_f, +)$ is an HX-group on $(R, +)$ and from the condition (1) it follows that $(\mathcal{R}_f, \cdot)$ is a semigroup. Since the multiendomorphism f fulfills the conditions of Theorem 3.3, then for arbitrary $f(x), f(y) \in \mathcal{R}_f$ it holds:

$$f(x) \cap f(y) \neq \emptyset \Longrightarrow f(x) = f(y).$$

Thus, the family $\mathcal{R}_f \subseteq P_0(R)$ fulfils the conditions of Theorem 3.1 and for that reason $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.

Corollary 3.6. *Let $f : R \to P_0(R)$ be a multiendomorphism of a ring $(R, +, \cdot)$ such that $f(0)$ is a subgroup of a group $(R, +)$. Then, the structure $(\mathcal{R}_f^*, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.*

Proof. As for all $x, y \in R$ it holds

$$f(x) \cdot f(y) = f(xy),$$

it is easy to see that the multiendomorphism f fulfils the conditions of the previous theorem.

Corollary 3.7. *Let $(R, +, \cdot)$ be a ring and $f : R \to P_0(R)$ multiendomorphism of a group $(R, +)$, such that $f(0)$ is an ideal of a ring R, which satisfies the condition $f(0) \cdot f(0) = f(0)$. If $\bigcup_{x \in R} f(x) = R$, then the structure $(\mathcal{R}_f^* = R, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.*

Proof. Let $x, y \in R$. From Corollary 3.4, for arbitrary $a \in f(x)$ and $b \in f(y)$, it holds:

$$f(x) = a + f(0) \quad \text{and} \quad f(y) = b + f(0)$$

and thus:

$$\begin{aligned} f(x)\cdot f(y) &= (a+f(0))(b+f(0)) = ab+f(0)b+af(0)+f(0)\cdot f(0) \\ &\subseteq ab+f(0)+f(0)+f(0) = ab+f(0). \end{aligned}$$

On the other hand,

$$ab+f(0) = ab+f(0)\cdot f(0) \subseteq ab+f(0)b+af(0)+f(0)\cdot f(0)$$

since $0\in a\cdot f(0)$ and $0\in f(0)b$. Thus, $f(x)\cdot f(y) = ab+f(0)$. From $\bigcup\limits_{x\in R} f(x) = R$, it follows that there exists $z\in R$ such that $ab\in f(z)$, and by Corollary 3.4, we have:

$$f(z) = ab+f(0).$$

Therefore,

$$f(x)\cdot f(y) = ab+f(0) = f(z).$$

From Theorem 3.5, it follows that $(\mathcal{R}_f^* = R, \widehat{\oplus}, \widehat{\odot})$ is a hyperring.

Example 3.8. Let $(R,+,\cdot) = (Z_6, +_6, \cdot_6)$ and $f: Z_6 \to P_0(Z_6)$ be a multimapping defined by $f(x) = (3\cdot_6 x) +_6 A$, where $A = \{0,2,4\}$. Then:

$$\begin{aligned} f(x+y) &= (3\cdot_6(x+_6 y)) +_6 A = (3\cdot_6 x) +_6 (3\odot_6 y) +_6 A \\ &= ((3\cdot_6 x)+_6 A) +_6 ((3\cdot_6 y)+_6 A) = f(x) +_6 f(y), \end{aligned}$$

since $A +_6 A = A$.

2) $f(0) = A$ and A is obviously an ideal of a ring $(Z_6, +_6, \cdot_6)$.

3) $A\cdot A = A$.

4) $\bigcup\limits_{x\in Z_6} f(x) = Z_6$.

Thus, f satisfied the conditions of the previous corollary. Therefore, the family $\mathcal{R}_f = \{f(x) \mid x\in Z_6\}$ generates the hyperring $(Z_6, \widehat{\oplus}, \widehat{\odot})$.

Generally, if A is an ideal of a ring $(R,+,\cdot)$ such that $A\cdot A = A$, then for arbitrary homomorphism $h: R\to R$ of additive group $(R,+)$, such that $h(R)+A = R$, there exists a multiendomorphism $f: R\to P_0(R)$ of a group $(R,+)$, defined by $f(x) = h(x)+A$, that satisfied the conditions of Theorem 3.5.

Theorem 3.9. *Let $f: R\to P_0(R)$ be a multiendomorphism of a ring $(R,+,\cdot)$ such that $f(0)$ is subgroup of a group $(R,+)$. Then, $f(0)$ is an ideal of a ring $(f(R),+,\cdot)$ and there exists a homomorphism $\bar{f}$ of a ring R into the quotient ring $f(R)/f(0)$ such that $f(x) = \bar{f}(x)$ for all $x\in R$. (In a quotient ring $f(R)/f(0)$ we have ordinary operations $\oplus$ and $\odot$ defined by*

$$\begin{aligned} (a+f(0))\oplus(b+f(0)) &= a+b+f(0), \\ (a+f(0))\odot(b+f(0)) &= ab+f(0)). \end{aligned}$$

Proof. Let us prove that $f(R)$ is a subring of a ring $(R, +, \cdot)$. Let $a, b \in f(R)$. Then, there exist $x, y \in R$ such that $a \in f(x)$, $b \in f(y)$. Thus,

$$a + b \in f(x) + f(y) = f(x + y) \subseteq f(R) \text{ and } ab \in f(x) \cdot f(y) = f(xy) \subseteq f(R).$$

If $a \in f(x)$, then

$$(-a) \in -(f(x)) \overset{Th.2.11}{=\!=} f(-x) \subseteq f(R).$$

Therefore, $f(R)$ is a subring of a ring R.

Let us prove that $f(0)$ is an ideal of a ring $f(R)$. Since $f(0)$ is a subgroup of a group $(f(R), +)$, it is sufficient to notice that for arbitrary $a \in f(R)$ it holds:

$$a \cdot f(0) \subseteq f(R) \cdot f(0) = f(0)$$

and

$$f(0) \cdot a \subseteq f(0) \cdot f(R) = f(0).$$

As by the Theorem 3.3 for arbitrary $x, y \in R$ it holds $f(x) \cap f(y) \neq \emptyset \Longrightarrow f(x) = f(y)$, we have that the mapping $\bar{f} : R \to f(R)/f(0)$ defined by:

$$\bar{f}(x) = f(x)$$

is well defined. (It is clear that, for arbitrary $x \in R$, from Corollary 3.4 it holds $f(x) = a + f(0)$ for arbitrary $a \in f(x) \subseteq f(R)$ and, therefore, $f(x) \in f(R)/f(0)$).

Let us prove that $\bar{f}$ is a homomorphism. Let $x, y \in R$. Then it holds:

$$\bar{f}(x + y) = f(x + y) = f(x) + f(y) = (*)$$

Let $a \in f(x)$ and $b \in f(y)$. Then:

$$\begin{aligned} (*) = (a + f(0)) + (b + f(0)) &= a + b + f(0) = (a + f(0)) \oplus (b + f(0)) \\ &= f(x) \oplus f(y) = \bar{f}(x) \oplus \bar{f}(y). \end{aligned}$$

Thus, $\bar{f}(x + y) = \bar{f}(x) \oplus \bar{f}(y)$.

Similarly,

$$\begin{aligned} \bar{f}(xy) = f(xy) &= f(x) \cdot f(y) = (a + f(0)) \cdot (b + f(0)) \\ &= ab + f(0) \cdot b + a \cdot f(0) + f(0) \cdot f(0) \\ &= ab + f(0) \cdot b + a \cdot f(0) + f(0 \cdot 0) \\ &= ab + f(0) \cdot b + a \cdot f(0) + f(0). \end{aligned}$$

Since, $f(0) \cdot b + a \cdot f(0) \subseteq f(0)$, it holds:

$$f(0) \cdot b + a \cdot f(0) + f(0) = f(0).$$

Therefore,

$$\bar{f}(xy) = ab + f(0) = (a + f(0)) \odot (b + f(0)) = f(x) \odot f(y) = \bar{f}(x) \odot \bar{f}(y).$$

Theorem 3.10. *Let $(R, +, \cdot)$ be a ring and let $f : R \to P_0(R)$ be a multiendomorphism of a group $(R, +)$ such that $f(0)$ is an ideal of a ring R which satisfies the conditions:*

1) $f(0) \cdot f(0) = f(0)$

2) $\bigcup\limits_{x \in R} f(x) = R$.

Then the family $\mathcal{R}_f = \{f(x) \mid x \in R\}$ *generates the hyperring* $(R_f^* = R, \widehat{\oplus}, \widehat{\odot})$ *and the mapping* $\Phi : R \to P_0(R)$ *defined by* $\Phi(x) = x + f(0)$ *is a good multihomomorphism of a ring* $(R, +, \cdot)$ *into a hyperring* $(R, \widehat{\oplus}, \widehat{\odot})$.

Proof. Let $x, y \in R$. Then:

$$\Phi(x + y) = x + y + f(0)$$

and

$$\Phi(x)\widehat{\oplus}\Phi(y) = (x + f(0))\widehat{\oplus}(y + f(0)) = \bigcup_{\substack{u \in x+f(0) \\ v \in y+f(0)}} u\widehat{\oplus}v = x + y + f(0).$$

Indeed, as $\bigcup\limits_{z \in R} f(z) = R$, it follows by Theorem 3.3 that there exists only one element $f(a)$ of the family $\mathcal{R}_f = \{f(z) \mid z \in R\}$ such that $x \in f(a)$ and, by Corollary 3.4, it holds $f(a) = x + f(0)$. Similarly, there exists only one element $f(b)$ of the family $\mathcal{R}_f$ such that $f(b) = y + f(0)$. Thus, for arbitrary $u \in f(a)$ and $v \in f(b)$, we have

$$u\widehat{\oplus}v = \bigcup_{\substack{u \in f(a') \\ v \in f(b')}} f(a') + f(b') = f(a) + f(b) = x + y + f(0).$$

Similarly,

$$\Phi(xy) = xy + f(0),$$

and

$$\Phi(x)\widehat{\odot}\Phi(y) = \bigcup_{\substack{u \in x+f(0) \\ v \in y+f(0)}} u\widehat{\odot}v \stackrel{(**)}{=} \bigcup_{\substack{u \in x+f(0) \\ v \in y+f(0)}} uv + f(0) \stackrel{(***)}{=} xy + f(0).$$

The equality $(**)$ is valid since, for $u \in x + f(0) = f(a)$ and $v \in y + f(0) = f(b)$, it holds:

$$\begin{aligned} u\widehat{\odot}v &= \bigcup_{\substack{u \in f(a') \\ v \in f(b')}} f(a') \cdot f(b') = f(a) \cdot f(b) \\ &= (u + f(0))(v + f(0)) \\ &= uv + f(0) \cdot v + u \cdot f(0) + f(0) \cdot f(0) \\ &= uv + f(0) \cdot v + u \cdot f(0) + f(0) \\ &= uv + f(0). \end{aligned}$$

The equality $(***)$ is valid since $x \in x + f(0),\ y \in y + f(0)$, and thus:

$$xy + f(0) \subseteq \bigcup_{\substack{u \in x+f(0) \\ v \in y+f(0)}} uv + f(0),$$

while, for arbitrary $u \in x + f(0)$ and $v \in y + f(0)$, it holds:

$$u + f(0) = x + f(0) \quad \text{and} \quad v + f(0) = y + f(0)$$

and thus:

$$uv + f(0) = (u + f(0))(v + f(0)) = (x + f(0))(y + f(0)) = xy + f(0).$$

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