LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-Paley operator and the functions b_j on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where the functions b_j belong to Lipschiz space.

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1. Introduction

We know, the commutator $[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ when T is the Calderón-Zygmund operator and $b \in BMO(R^n)$. Janson and Paluszynski study the commutator for the Triebel-Lizorkin space and the case $b \in Lip_\beta(R^n)$, where $Lip_\beta(R^n)$ is the homogeneous Lipschitz space. Chanillo (see [2]) proves a similar result when T is replaced by the fractional operators. The main purpose of this paper is to discuss the boundedness of Littlewood-Paley multilinear commutator generated by Littlewood-Paley operator and Lipschiz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Q will denote a cube of R^n with side parallel to the axes.

Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$ denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)$ $(0 < p \le 1)$ has the atomic decomposition characterization (see [11], [16], [17]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space.

The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1. (see [15]) *For* $0 < \beta < 1,\ 1 < p < \infty,$ *we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left\|\sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx\right\|_{L^p} \\ &\approx \left\|\sup_{\cdot\in Q} \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-c|dx\right\|_{L^p}. \end{aligned}$$

Lemma 2. (see [15]) *For* $0 < \beta < 1,\ 1 \le p \le \infty,$ *we have*

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{1/p}. \end{aligned}$$

Lemma 3. (see [2]) *For* $1 \le r < \infty$ *and* $\beta > 0,$ *let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}}\int_Q |f(y)|^r dy\right)^{1/r},$$

suppose that $r < p < n/\beta,$ *and* $1/q = 1/p - \beta/n,$ *then*

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4. (see [5]) *Let* $Q_1 \subset Q_2,$ *then*

$$|f_{Q_1} - f_{Q_2}| \le C||f||_{\dot{\Lambda}_\beta}|Q_2|^{\beta/n}.$$

Definition 1. Let $0 < p,\ q < \infty$, $\alpha \in R$, $B_k = \{x \in R^n, |x| \le 2^k\},\ A_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{Ak}$ for $k \in \mathbf{Z}$.

1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 5. (see [6], [14]) *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}} (or\ ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 4. Let $0 < \delta < n$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

1) $\int_{R^n} \psi(x)dx = 0$,
2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

Let m be a positive integer and $b_j (1 \le j \le m)$ be the locally integrable function, set $\vec{b} = (b_1, \cdots, b_m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy,$$

and

$$\psi_t(x) = t^{-n+\delta}\psi(x/t)$$

for $t > 0$. Set $F_t(f) = \psi_t * f$. We also define that

$$g_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley g function (see [17]).

Let H be the space

$$H(R^n) = \{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\},$$

then, for each fixed $x \in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}(f)(x) = ||F_t(f)(x)|| \text{ and } g_{\psi,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $g_{\psi,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4], [7-10], [12], [15]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set

$$||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$$

and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta} = ||b_{\sigma(1)}||_{Lip_\beta} \cdots ||b_{\sigma(j)}||_{Lip_\beta}$.

Lemma 6. (see [10]) *Let $0 < \beta \le 1, 0 < \delta < n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $b \in Lip_\beta(R^n)$. Then $g_{\psi,\delta}^b$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

3. Theorems and proofs

Theorem 1. *Let $0 < \delta < n$, $0 < \beta < min(1, \varepsilon/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in Lip_\beta(R^n)$ for $1 \le j \le m$ and $g_{\psi,\delta}^{\vec{b}}$ be the multilinear commutator of Littlewood-Paley operator as in Definition 4. Then*

a) *$g_{\psi,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{m\beta,\infty}(R^n)$ for $1<p<n/\delta$ and $1/p-1/q=\delta/n$.*

b) *$g_{\psi,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for $1/p - 1/q = m\beta + \delta/n$ and $1/p > m\beta + \delta/n$.*

Proof. (a). Fixed a cube $Q = (x_0, l)$ and $\tilde{x} \in Q$, see [10] when $m = 1$.

Consider now the case $m \ge 2$. Set

$$\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q),$$

where

$$(b_j)_Q = |Q|^{-1} \int_Q b_j(y)dy, \ 1 \le j \le m.$$

Writing $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}, f_2 = f\chi_{R^n \setminus 2Q}$, we have

$$\begin{aligned}
F_t^{\vec{b}}(f)(x) &= \int_{R^n} (b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y))\psi_t(x-y)f(y)dy \\
&= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma \int_{R^n} (b(y) - \vec{b}_Q)_{\sigma^c}\psi_t(x-y)f(y)dy \\
&= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c} f)(x),
\end{aligned}$$

then

$$|g_{\psi,\delta}^{\vec{b}}(f)(x) - g_{\psi,\delta}(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|$$
$$\leq ||F_t^{\vec{b}}(f)(x) - F_t(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)||$$
$$\leq ||(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(x)||$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c}f)(x)||$$
$$+||F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)||$$
$$+||F_t((b_1 - (b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||$$
$$= I_1(x) + I_2(x) + I_3(x) + I_4(x),$$

so,

$$\begin{aligned}
&\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q |g_{\psi,\delta}^{\vec{b}}(f)(x) - g_{\psi,\delta}((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\
&\leq \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_1(x)dx + \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_2(x)dx \\
&+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_3(x)dx + \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_4(x)dx \\
&= I + II + III + IV.
\end{aligned}$$

For I, by using Lemma 2, we have

$$\begin{aligned}
I &\leq \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\sup_{x\in Q}|b_1(x) - (b_1)_Q|\cdots|b_m(x) - (b_m)_Q|\int_Q |g_{\psi,\delta}(f)(x)|dx \\
&\leq C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{m\beta}{n}}\int_Q |g_{\psi,\delta}(f)(x)|dx \\
&\leq C||\vec{b}||_{Lip_\beta}M(g_{\psi,\delta}(f))(\tilde{x}).
\end{aligned}$$

For II, taking $1 < r < p < q < n/\delta$, $1/q'+1/q = 1$, $1/s'+1/s = 1$, $1/q = 1/p-\delta/n$, $ps = r$ by using the Hölder's inequality and the boundedness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Lemma 2, we get

$$\begin{aligned}
II &\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma||g_{\psi,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c}f)(x)|dx \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'} \\
&\qquad\times\left(\frac{1}{|Q|}\int_{R^n}|g_{\psi,\delta}((\vec{b} - \vec{b}_Q)_{\sigma^c}f\chi_Q)(x)|^q dx\right)^{1/q}
\end{aligned}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{|\sigma|\beta/m}\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}f\chi_Q|^p dx\right)^{1/p}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps}$$

$$\times\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{ps'}\right)^{1/ps'}\left(\frac{1}{|Q|^{1-\frac{\delta ps}{n}}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{|\sigma|\beta/n}||\vec{b}_{\sigma^c}||_{Lip_\beta}|Q|^{|\sigma^c|\beta/n}M_{r,\delta}(f)(\tilde{x})$$

$$\leq C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).$$

For III, we choose $1<r<p<q<n/\delta$, $0<\delta<n$, $1/q=1/p-\delta/n$, $r=ps$, by the boundness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/s+1/s'=1$, we get

$$\begin{aligned}
III &= \frac{1}{|Q|^{1+m\beta/n}}\int_Q|g_{\psi,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|dx\\
&\leq C\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_{R^n}|g_{\psi,\delta}(\prod_{j=1}^m(b_j(y)-(b_j)_Q)f\chi_Q)(x)|^q dx\right)^{1/q}\\
&\leq C\frac{1}{|Q|^{m\beta/n}}\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^m(b_j(y)-(b_j)_Q)|^p|f\chi_Q|^p dx\right)^{1/p}\\
&\leq C\frac{1}{|Q|^{m\beta/n}}|Q|^{(-1/q)+1/ps'-(1-(\delta ps/n)/ps)}\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m(b_j(y)-(b_j)_Q)|^{ps'}dx\right)^{1/ps'}\\
&\quad\times\left(\frac{1}{|Q|^{1-\delta ps/n}}\int_Q|f(x)|^{ps}dx\right)^{1/ps}\\
&\leq C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}$$

For IV, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, by Lemma 4 and the condition on ψ, we have

$$||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||$$

$$\leq\left[\int_0^\infty\left(\int_{(2Q)^c}|\psi_t(x-y)-\psi_t(x_0-y)||f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}$$

$$\le C\left[\int_0^\infty\left(\int_{(2Q)^c}\frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon-\delta}}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}$$

$$\le C\int_{(2Q)^c}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon-\delta)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy$$

$$\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon-\delta)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy$$

$$\le C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|\prod_{j=1}^m(|b_j(y)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}}-(b_j)_Q|)dy$$

$$\le C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{\delta/n}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x})$$

$$\le C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M_{r,\delta}(f)(\tilde{x}),$$

thus

$$IV\le C||\vec{b}||_{Lip_\beta}M_{r,\delta}(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $x\in Q$, we obtain

$$||g_{\psi,\delta}^{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}}\le C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This complete the proof of (a).

(b) By some argument as in proof of (a), we have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q|g_{\psi,\delta}^{\vec{b}}(f)(x)-g_{\psi,\delta}(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx\\ &\le\frac{1}{|Q|}\int_Q I_1(x)dx+\frac{1}{|Q|}\int_Q I_2(x)dx+\frac{1}{|Q|}\int_Q I_3(x)dx+\frac{1}{|Q|}\int_Q I_4(x)dx\\ &\le C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x}))+M_{m\beta+\delta,r}(f)(\tilde{x})),\end{aligned}$$

thus

$$(g_{\psi,\delta}^{\vec{b}}(f))^\#\le C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x}))+M_{m\beta+\delta,r}(f)(\tilde{x})).$$

By using Lemma 3 and the boundedness of $g_{\psi,\delta}$, we have

$$\begin{aligned}||g_{\psi,\delta}^{\vec{b}}(f)||_{L^q}&\le C||(g_{\psi,\delta}^{\vec{b}}(f))^\#||_{L^q}\\ &\le C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(g_{\psi,\delta}(f)(\tilde{x}))+M_{m\beta+\delta,r}(f)(\tilde{x}))||_{L^q}\\ &\le C||\vec{b}||_{Lip_\beta}||f||_{L^p}.\end{aligned}$$

This complete the proof of (b) and the theorem.

Theorem 2. *Let* $0 < \delta < n$, $0 < \beta + \delta/m < \min(\gamma/m, 1/2m)$, $n/(n+\beta+\delta/m) < p \leq 1$, $1/q = 1/p - (m\beta+\delta)/n$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \leq j \leq m$. *Then* $g_{\psi,\delta}^{\vec{b}}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||g_{\psi,\delta}^{\vec{b}}(a)||_{L^q} \leq C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \leq |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p-1)]$.

When $m = 1$, see [10]. Now consider the case $m \geq 2$. Write

$$\begin{aligned} ||g_{\psi,\delta}^{\vec{b}}(a)(x)||_{L^q} &\leq \left(\int_{|x-x_0|\leq 2r} |g_{\psi,\delta}^{\vec{b}}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |g_{\psi,\delta}^{\vec{b}}(a)(x)|^q dx\right)^{1/q} \\ &= I + II. \end{aligned}$$

For I, choose $1 < p_1 < n/(m\beta+\delta)$ and q_1 such that $1/q_1 = 1/p_1 - m\beta + \delta/n$. By the boundednss of $g_{\psi,\delta}^{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$ (see Theorem 1), we get

$$\begin{aligned} I &\leq C||g_{\psi,\delta}^{\vec{b}}(a)||_{L^{q_1}}^q |Q(x_0, 2r)|^{1-q/q_1} \leq C||a||_{L^{p_1}}^q ||\vec{b}||_{Lip_\beta} |Q|^{1-q/q_1} \\ &\leq C||\vec{b}||_{Lip_\beta} |Q|^{-q/p+q/p_1+1-q/q_1} \leq C||\vec{b}||_{Lip_\beta}. \end{aligned}$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \neq 0$. We get

$$|F_t^{\vec{b}}(a)(x)| \leq |(b_1(x)-b_1(x_0))\cdots(b_m(x)-b_m(x_0))\int_B (\psi_t(x-y)-\psi_t(x-x_0))a(y)dy|$$

$$\begin{aligned} &+\sum_{j=1}^m \sum_{\sigma\in C_j^m} |(b(x)-b(x_0))_{\sigma^c} \int_B (b(y)-b(x_0))_\sigma \psi_t(x-y)a(y)dy| \\ &\leq C||\vec{b}||_{Lip_\beta} |x-x_0|^{m\beta} \cdot \int_B |\psi_t(x-y)-\psi_t(x-x_0)||a(y)|dy \\ &+C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \int_B |y-x_0|^{\tau'\beta} |\psi_t(x-y)||a(y)|dy \\ &\leq C||\vec{b}||_{Lip_\beta} \frac{|x-x_0|^{m\beta} t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}} \int_B |x_0-y|^\varepsilon |a(y)|dy \\ &+C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \frac{t}{(t+|x-x_0|)^{n+1-\delta}} \int_B |y-x_0|^{\tau'\beta} |a(y)|dy \\ &\leq C||\vec{b}||_{Lip_\beta} \frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}} \cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})} \\ &+C||\vec{b}||_{Lip_\beta} \frac{t}{(t+|x-x_0|)^{n+1-\delta}} \cdot r^{m\beta+n(1-\frac{1}{p})}, \end{aligned}$$

thus

$$
\begin{aligned}
|g_{\psi,\delta}^{\vec{b}}(a)(x)| &\leq C||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})} \\
&+ C||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\left(\frac{t}{(t+|x-x_0|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot r^{m\beta+n(1-\frac{1}{p})} \\
&\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{-n+\delta}\cdot r^{m\beta+n(1-\frac{1}{p})},
\end{aligned}
$$

so,

$$
II \leq C||\vec{b}||_{Lip_\beta}\cdot r^{m\beta+n(1-\frac{1}{p})}\left(\int_{|x-x_0|>2r}|x-x_0|^{-nq+q\delta}dx\right)^{1/q}\leq C||\vec{b}||_{Lip_\beta}.
$$

This complete the proof of Theorem 2.

Theorem 3. *Let* $0<\beta\leq 1$, $0<\delta<n$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2=m\beta+\delta/n$, $n(1-1/q_1)\leq\alpha<n(1-1/q_1)+\beta+\delta/m$, $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in Lip_\beta(R^n)$ *for* $1\leq j\leq m$. *Then* $g_{\psi,\delta}^{\vec{b}}$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

Proof. By Lemma 5, let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, $supp\, a_j\subset B_j=B(0,2^j)$, a_j be a central $(\alpha,q)-$atom, and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$. We have

$$
\begin{aligned}
||g_{\psi,\delta}^{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha,p}}^p &\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|||g_{\psi,\delta}^{\vec{b}}(a_j)\chi_k||_{L^{q^2}}\right)^p \\
&+ C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|||g_{\psi,\delta}^{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p \\
&= I+II.
\end{aligned}
$$

For II, by the boundedness of $g_{\psi,\delta}^{\vec{b}}$ on (L^{q_1},L^{q_2}), we have

$$
\begin{aligned}
II &\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|||a_j||_{L^{q_1}}\right)^p \\
&\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\cdot 2^{-j\alpha}\right)^p \\
&\leq C||\vec{b}||_{Lip_\beta}^p\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{(k-j)\alpha p},\ 0<p\leq 1\\ \displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p/2}\right)\left(\sum_{j=k-1}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'},\ 1<p<\infty\end{cases}
\end{aligned}
$$

$$\leq C||\vec{b}||_{Lip_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

For I, when $m=1$, we have

$$\begin{aligned}
|F_t^{b_1}(a_j)(x)| &\leq \left|(b_1(x)-b_1(0))\int_{B_j}(\psi_t(x-y)-\psi_t(x))a_j(y)dy\right| \\
&+ \left|\int_{B_j}\psi_t(b_1(y)-b_1(0))a_j(y)dy\right| \\
&\leq C||b_1||_{Lip_\beta}\left[\int_{B_j}\frac{|x|^\beta|y|^\varepsilon t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot|a_j(y)|dy\right. \\
&+ \left.\int_{B_j}\frac{t|y|^\beta}{(t+|x-y|)^{n+1-\delta}}\cdot|a_j(y)|dy\right] \\
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right. \\
&+ \left.\frac{t}{(t+|x|)^{n+1-\delta}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right] \\
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right. \\
&+ \left.\frac{t}{(t+|x|)^{n+1-\delta}}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right],
\end{aligned}$$

thus

$$\begin{aligned}
g_{\psi,\delta}^{b_1}(a_j)(x) &\leq C||b_1||_{Lip_\beta}\left[\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\right)^2\right)^{1/2}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right. \\
&+ \left.\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right] \\
&\leq C||b_1||_{Lip_\beta}\left[|x|^{-(n+\varepsilon-\delta)}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right] \\
&\leq C||b_1||_{Lip_\beta}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}$$

from that we have

$$\begin{aligned}
||g_{\psi,\delta}^{b_1}(a_j)\chi_k||_{L^{q_2}} &\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\left(\int B_k|x|^{-nq_2+q_2\delta}dx\right)^{1/q_2} \\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot 2^{-kn(1-\frac{1}{q_2})+k\delta} \\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}$$

so,

$$I \leq C||b_1||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^p$$

$$
\leq C||b_1||^p_{Lip_\beta}\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right) & \\ \displaystyle\quad\times\left(\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, & 1<p<\infty\end{cases}
$$

$$
\leq C||b_1||^p_{Lip_\beta}\begin{cases}\displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \displaystyle\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{\frac{p}{2}[(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)]}, & 1<p<\infty\end{cases}
$$

$$
\leq C||b_1||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
$$

Then

$$
||g^{b_1}_{\psi,\delta}(f)||_{\dot{K}^{\alpha,p}_{q_2}}\leq C||b_1||_{Lip_\beta}(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.
$$

When $m>1$, we have

$$
\begin{aligned}
|F^{\vec{b}}_t(a_j))(x)| &\leq |(b_1(x)-b_1(0))\cdots(b_m(x)-b_m(0))\int_{B_j}(\psi_t(x-y)-\psi_t(x))a_j(y)dy|\\
&\quad+\sum_{j=1}^{\infty}\sum_{\sigma\in C^m_j}|(b(x)-b(0))_{\sigma^c}\int_{B_j}(b(y)-b(0))_\sigma\psi_t(x-y)a_j(y)dy|\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\int_{B_j}|\psi_t(x-y)-\psi_t(x)||a_j(y)|dy\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\int_{B_j}|y|^{\tau'\beta}|\psi_t(x-y)||a_j(y)|dy\\
&\leq C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon-\delta}}\int_{B_j}|y|^{\varepsilon}|a_j(y)|dy\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1-\delta}}\int_{B_j}|y|^{\tau'\beta}|a_j(y)|dy\\
&\leq C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon-\delta}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\\
&\quad+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1-\delta}}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}
$$

thus

$$
\begin{aligned}
g^{\vec{b}}_{\psi,\delta}(a_j)(x) &= \left(\int_0^\infty|F^{\vec{b}}_t(a_j)(x)|^2\frac{dt}{t}\right)^{1/2}\\
&\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}
\end{aligned}
$$

$$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dt}{t}\right)^{1/2}$$

$$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}$$

$$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-n+\delta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}$$

$$\leq C||\vec{b}||_{Lip_\beta}|x|^{-n+\delta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)},$$

then

$$\begin{aligned}||g^{\vec{b}}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} &\leq C||\vec{b}||_{Lip_\beta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_{B_j}|x|^{-nq_2+q_2\delta}dx\right)^{1/q_2}\\ &\leq C||\vec{b}||_{Lip_\beta}\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]},\end{aligned}$$

so,

$$\begin{aligned}I &\leq C||\vec{b}||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^p\\
&\leq C||\vec{b}||^p_{Lip_\beta}\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(m\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \displaystyle\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}\left[j\left(m\beta+n\left(1-\frac{1}{q_1}\right)-\alpha\right)-k\left(m\beta+n\left(1-\frac{1}{q_1}\right)\right)\right]}\right) & \\ \displaystyle\quad\times\left(\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, & 1<p<\infty\end{cases}\\
&\leq C||\vec{b}||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.\end{aligned}$$

From I and II, we have

$$||g^{\vec{b}}_{\psi,\delta}(f)||\leq C||\vec{b}||_{Lip_\beta}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.$$

This completes the proof of Theorem 3.

Theorem 4. *Let* $0<\beta<\min(\gamma/m,1/2m)$, $0<p\leq 1$, $1<q_1,q_2<\infty$, $0<\delta<n$, $1/q_2=1/q_1-(m\beta+\delta)/n$, $\vec{b}=(b_1,\cdots,b_m)$ *with* $b_j\in Lip_\beta(R^n)$ *for* $1\leq j\leq m$. *Then* $g^{\vec{b}}_{\psi,\delta}$ *maps* $H\dot{K}^{n(1-1/q_1)+\beta+\delta/m,p}_{q_1}(R^n)$ *continuously into* $W\dot{K}^{n(1-1/q_1)+\beta+\delta/m,p}_{q_2}(R^n)$.

Proof. We write

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k,$$

where each a_k is a central $(n(1-1/q_1)+\beta+\delta/m, q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p<\infty$. Write

$$
\begin{aligned}
&||g_{\psi,\delta}^{\vec{b}}||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}}\\
&\leq \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}\left|\left\{x\in E_l:\left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=l-3}^{\infty}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\
&+\sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}\left|\left\{x\in E_l:\left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)(x)\right|>\lambda/2\right\}\right|^{p/q_2}\right\}^{1/p}\\
&=G_1+G_2.
\end{aligned}
$$

By the (L^{q_1},L^{q_2}) boundedness of $g_{\psi,\delta}^{\vec{b}}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$
G_1^p\leq C\sum_{l=-\infty}^{\infty}2^{lp(n(1-1/q_1)+\beta+\delta/m)}\left|\left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{l-3}^{\infty}\lambda_k a_k\right)(x)\chi_l\right|\right|_{q_2}^p\leq C||\vec{b}||_{Lip_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.
$$

To estimate G_2, let us now use the estimate

$$
|g_{\psi,\delta}^{\vec{b}}(a_k)| \ \leq \ C||\vec{b}||_{Lip_\beta}|x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha},
$$

which we get in the proof of Theorem 3.

Note that when $x\in E_l$, $\alpha=n(1-1/q_1)+\beta+\delta/m$,

$$
\begin{aligned}
\lambda<\sum_{k=-\infty}^{l-4}|\lambda_k||g_{\psi,\delta}^{\vec{b}}(a_k)| &\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k||x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha}\\
&\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k||2^l|^{\delta-n}\sum_{k=-\infty}^{l-4}(2^k)^{m\beta+n(1-1/q_1)-\alpha}\\
&\leq C||\vec{b}||_{Lip_\beta}\sum_{k=-\infty}^{l-4}|\lambda_k|(2^l)^{((m-1)\beta+\delta-n-\delta/m)}\\
&\leq C||\vec{b}||_{Lip_\beta}2^{l((m-1)\beta+\delta-n-\delta/m)}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},
\end{aligned}
$$

for $\lambda>0$, let l_λ be the maximal positive integer satisfying

$$
2^{l_\lambda(n+\delta/m-(m-1)\beta-\delta)}\leq C||\vec{b}||_{Lip_\beta}\lambda^{-1}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},
$$

then if $l>l_\lambda$, we have

$$
|\{x\in E_l:|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)|>\lambda/2\}|=0.
$$

So, we obtain

$$\begin{aligned} G_2 &\le \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\beta+\delta/m)p} (2^l)^{np/q_2} \right\}^{1/p} \\ &\le \sup_{\lambda>0} \lambda \left\{ \sum_{l=-\infty}^{l_\lambda} (2^l)^{(n-(m-1)\beta-\delta)} \right\}^{1/p} \\ &\le \sup_{\lambda>0} \lambda 2^{l_\lambda (n-(m-1)\beta-\delta)} \le C ||\vec{b}||_{Lip_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}. \end{aligned}$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$||g_{\psi,\delta}^{\vec{b}}(f)||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}} \le C ||\vec{b}||_{Lip_\beta} \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}.$$

Theorem 4 follows by taking the infimum over all central atomic decompositions.

References

[1] J. Alvarez, R. J. Babgy, D. S. Kurtz and C. Pérez, C., *Weighted estimates for commutators of linear operators*, Studia Math., 104 (1993), 195-209.

[2] Chanillo, S., *A not on commutators*, Indiana Univ. Math. J., 31 (1982), 7-16.

[3] Chen, W,G., *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, 16 (2000), 613-626.

[4] R. Coifman, R. Rochberg and G. Weiss, G., *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., 103 (1976), 611-635.

[5] Devore, R.A. and Sharply, R.C., *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., 47(1984).

[6] J. Garcia-Cuerva and M. J. L. Herrero, J.L., *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc., 69 (1994), 605-628.

[7] Janson, S., *Mean Oscillation and commutators of singular integral operators*, Ark. Math., 16 (1978), 263-270.

[8] Liu, L.Z., *Boundedness of multilinear operator on Triebel-Lizorkin spaces*, Inter. J. of Math. and Math. Sci., 5 (2004), 259-271.

[9] Liu, L.Z., *The continuity of commutators on Triebel-Lizorkin spaces*, Integral Equations and Operator Theory, 49 (2004), 65-76.

[10] Liu, L.Z., *Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces*, Extracta Math., 19 (2) (2004), 243-255.

[11] Lu, S.Z., *Four lectures on real H^p spaces*, World Scientific, River Edge, NI, 1995.

[12] Lu, S.Z., Q. Wu, Q. and Yang, D.C., *Boundedness of commutators on Hardy type spaces*, Sci. in China (Ser. A), 45 (2002), 984-997.

[13] Lu, S.Z. and Yang, D.C., *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (Ser. A), 38 (1995), 147-158.

[14] Lu, S.Z. and Yang, D.C., *The weighted Herz type Hardy spaces and its applications*, Sci. in China (Ser. A), 38 (1995), 662-673.

[15] Paluszynski, M., *Characterization of the Besov spaces via the commutator operator of Coifman*, Rochbeg and Weiss, Indiana Univ. Math. J., 44 (1995), 1-17.

[16] Stein, E.M., *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton, Princeton Univ. Press, 1993.

[17] Torchinsky, A., *Real variable methods in harmonic analysis*, Pure and Applied Math., 123, Academic Press, New York, 1986.

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