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Addressing learning loss through a mathematical web-app: insights from user behaviour

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Abstract: This study stems from the MaTEs project which aims to address summer learning loss in mathematics through a web-based app designed for pupils and their families. This paper presents some results focusing on the design and use of a web-based app aimed at supporting continuity of engagement with mathematical activities during the summer period. The study conceptualizes learning as a situated and mediated process extending beyond school contexts. The web app offers narrative-based problem-solving activities supported by adult mediation, integrating cognitive, metacognitive, and affective dimensions. Using log data from 315 users, the study adopts a quantitative approach based on Time Series K means clustering to identify patterns of behaviour over the summer period. Four distinct profiles emerge in terms of completion, continuity, and temporal distribution of activity. Findings highlight a general tendency toward non-uniform use, with activity concentrated at the end of the summer, but the largest group corresponding to a regular and sustained pattern of use. Notably, no significant differences are found across profiles in metacognitive reflections, suggesting that discontinuity of use cannot be simply explained by different levels of appreciation or perceived experience. The study sheds light on heterogeneous attitudes towards the summer homework and raises critical questions about how continuity, rather than intensity alone, supports meaningful mathematical learning beyond the classroom.

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1. Introduction

This paper presents the results of the MaTEs project[2], which investigates summer learning loss in mathematics, conceptualized as a mechanism similar to that observed during the COVID-19 pandemic, in terms of reduced school engagement and the increased role of families.

Continuity is a key element in ensuring effective learning: it enables students to consolidate knowledge and skills over time, progressively building stable cognitive structures. When this continuity is disrupted, disconnection arises between previous and subsequent learning experiences, with the risk that part of the acquired competencies may be forgotten or not properly consolidated. Thus, more generally, learning loss can be defined as the decline in skills and knowledge that occurs during periods when instructional activity is reduced or absent [34, 23, 32].

School discontinuity is closely related to learning loss. In particular, during extended breaks, students may experience a decrease in abilities, especially in areas such as mathematics and reading comprehension, which require consistent practice. This effect is often more pronounced among students from socio-economically disadvantaged backgrounds, where opportunities for informal learning during school interruptions are more limited [31].

This study, as part of a wider project aimed at addressing such a problem, focuses on educational intervention designed to counteract math learning loss, based on a web app for families and children. As the first investigation, this work does not assess learning achievements directly, but aims to investigate to what extent the use of the designed web-app can foster continuity supporting student engagement in informal learning activities during summer.

The conceptual background presented in this paper primarily informs the design of the MaTEs web-app, clarifying why the activities were structured around storytelling, modelling, adult mediation, and affective-metacognitive reflection. The empirical analysis reported here does not evaluate learning outcomes directly, but examines how the web-app was used over time, considering usage patterns as traces of engagement with the designed mathematical activities.

2. Conceptual background

The project is grounded in three interconnected strands of research: (i) learning loss and informal learning, (ii) storytelling and mathematical modelling, and (iii) affect and engagement in mathematics education. These perspectives are brought together to conceptualize learning as a situated, mediated, and multidimensional process that extends beyond formal classroom settings.

We use the term learning loss not to frame pupils or families from a deficit perspective, but to refer to the risk that discontinuities in school-based mathematical practices may reduce opportunities for sustained engagement, especially when access to informal learning resources and adult mediation is unevenly distributed.

2.1. Learning loss and informal learning

The phenomenon of *learning loss* has been widely documented in educational research, particularly in relation to prolonged interruptions of formal schooling, such as summer breaks or the COVID-19 pandemic. Studies consistently show that the absence of structured learning opportunities leads to a decline in students' academic achievement, especially in mathematics [4, 22]. Moreover, learning loss is not evenly distributed, as it tends to exacerbate existing

[2]MaTEs is the acronym of the Italian expression Matematica per Tutti in Estate, which can be translated as "Mathematics for Everyone in Summer".

inequalities due to differences in access to resources and support outside school [31].

Recent reviews have also pointed out that the notion of summer learning loss is not always theoretically stable and is described in the literature through different labels, such as summer slide, summer setback, or summer learning gap. For this reason, it is important to treat summer learning loss not as an automatic and uniform outcome of school interruption, but as a phenomenon connected to differences in opportunities, contexts, and forms of support available to students during the summer period [32]. From this perspective, the issue is not simply whether students "lose" knowledge, but how opportunities to participate in meaningful mathematical practices are maintained, transformed, or reduced across school and out-of-school contexts.

In this context, *informal learning environments*, such as home-based activities, play a crucial role. However, their effectiveness depends on the nature of the learning experiences they offer. Research suggests that simply extending school-like tasks into the home is insufficient; rather, informal learning should be structured in ways that promote active engagement, autonomy, and meaning-making [19]. This calls for the design of learning environments that can bridge formal and informal contexts, supporting continuity in students' learning trajectories while taking into account the specific characteristics of out-of-school settings. In such contexts, digital tools may therefore act not as autonomous learning environments, but as mediating devices that organize access to tasks, support adult guidance, and structure opportunities for children's participation in meaningful activities.

In this perspective, families – and particularly parents – may play a crucial role in shaping learning opportunities during school interruptions. Research on parental engagement highlights that home routines, homework support, and parents' mediation of learning activities significantly influence children's academic continuity and engagement [25]. In the context of summer learning loss, differences in family support and access to learning opportunities contribute to the unequal distribution of achievement trajectories across students [2].

During summer breaks, learning opportunities become increasingly dependent on family practices, parental mediation, and access to informal educational resources. Research shows that families differ significantly in the extent to which they maintain continuity with school-related activities through structured routines, shared learning practices, and autonomy - supportive forms of engagement [29, 27]. These differences are not neutral, as they influence both the quantity and the quality of students' engagement with learning activities, contributing to unequal educational opportunities outside school [8, 24]. In this perspective, summer learning differences should not be understood only as the consequence of reduced formal instruction, but also as the result of broader social and relational conditions shaping children's access to meaningful learning experiences [2, 41].

2.2. Storytelling, mathematical modelling and representation

Research in mathematics education frequently highlights a widespread disaffection toward the subject, often associated with negative attitudes rooted in the perception of mathematics as a set of rules to be memorized and mechanically applied. This procedural view limits students' engagement and reduces their ability to construct meaning from mathematical ideas. In response, storytelling has been proposed as a pedagogical approach capable of supporting meaning-making in mathematics learning. This perspective can be further enriched by the idea of mathematics curriculum as story by Dietiker [17], who argues that mathematical content can be interpreted as a narrative sequence in which mathematical objects, actions, settings, and plots unfold over time. From this point of view, storytelling in mathematics is not limited to adding an external narrative context to a task, but concerns the way mathematical ideas are progressively introduced, transformed, and connected within a meaningful sequence.

Storytelling is effective in education because narratives provide powerful structures for organizing information and interpreting human experience [42, 43]. Stories typically involve characters who face conflicts or challenges and attempt to resolve them. Unlike expository or scientific texts, stories engage both emotions and imagination. Narratives may orient the listener's feelings toward what is being told, making emotional involvement a central component of understanding. Consequently, storytelling fosters both cognitive and emotional engagement with mathematical content [44].

In mathematics education, stories can serve multiple functions: they may contextualize activities, introduce tasks, explain abstract concepts, or structure questions. In primary school, mathematical modelling is often introduced through word problems intended as simplified narratives. However, these problems are frequently stripped of narrative richness, encouraging superficial strategies such as keyword identification rather than genuine comprehension [43]. This reinforces the perception of mathematics as a meaningless procedure. To address this issue, Zan [42] emphasizes embedding mathematical questions within meaningful narrative contexts, where problems arise naturally from the story rather than being externally imposed, making tasks more authentic and closer to real-life situations.

The theoretical foundation for this approach lies in Bruner's distinction between narrative and paradigmatic thinking [11]. Narrative thinking focuses on human intentions, actions, and emotions, while paradigmatic thinking emphasizes logical structure and formal proof. Although distinct, these modes are complementary rather than hierarchical. In mathematical story-problems, mathematical reasoning should not be understood as exclusively paradigmatic, since it may also unfold through narrative structures as pupils interpret actions, goals, constraints, and consequences. Thus, narrative is not merely an external context, but can shape how mathematical relationships are noticed, connected, and justified [17].

Mathematical modelling involves meaning making through the connection between reality and mathematics. It is defined as the process of translating real-world situations into mathematical representations, working within mathematics, and interpreting and validating results in context [9]. The modelling cycle includes iterative steps such as understanding, simplifying/structuring, mathematizing, working mathematically, interpreting, validating, and presenting, though in practice it is rarely linear [10].

Bruner's distinction should not be directly mapped onto the distinction between reality and mathematics in modelling: the former concerns modes of sense-making, whereas the latter concerns the domains involved in the modelling process. In story-based modelling tasks, however, the two are connected, as narrative thinking supports pupils' understanding of the situation, while modelling guides the transition toward mathematical representations and their interpretation in context.

A key step is understanding the situation, which is essential for successful modelling [13]. However, students may struggle to move between narrative understanding and formal mathematical reasoning, either remaining at a descriptive level or focusing only on manipulation without interpretation [42]. Therefore, graphical representation plays a crucial role in understanding the situation and supporting mathematical modelling and meaning making. Drawing is increasingly understood not only as a representational tool but also as a form of mathematical thinking [37]. Throughout the modelling cycle, drawings evolve from situational representations that help students understand and structure a problem, to more abstract mathematical drawings that support reasoning and problem solving [36, 6]. They also aid in interpreting and validating results by connecting mathematical outcomes to real-world contexts. Research shows that drawing skills and accuracy are positively related to modelling performance, highlighting their importance as a bridge between reality and mathematics [36].

Overall, storytelling and modelling are complementary approaches for fostering meaningful learning in mathematics education. Storytelling provides an emotionally engaging entry point into mathematical ideas, while modelling offers a structured bridge between real-world contexts and formal mathematics. Their integration supports mathematical activity through the interaction of narrative sense-making, representation, modelling, imagination, experience, and logical reasoning.

2.3. Affect and engagement

Given the focus of the present study, we do not address the affective domain in its full complexity. Rather, we focus on three dimensions that are directly operationalized in the MaTEs web-app: task enjoyment, perceived performance, and perceived difficulty.

Research in mathematics education increasingly recognizes that learning and teaching mathematics involve not only cognitive processes but also a broad affective dimension, including emotions, motivation, attitudes, beliefs, values, and volition. These elements are deeply intertwined with thinking processes and significantly influence how students engage with mathematics. As Radford [35] argues, mathematics cannot be separated from affect, since thinking itself is always accompanied by emotional experience. From this perspective, affect is not an external component of learning but an intrinsic dimension of mathematical activity.

Given the complexity of the affective domain, research often focuses on specific constructs. In the MaTEs project, particular attention is given to attitudes toward mathematics. According to Deci and Ryan [15], positive attitudes are associated with greater autonomy, engagement, and improved learning outcomes. Di Martino and Zan [16] define attitudes as composed of three interconnected dimensions: interest in mathematics, perceived competence, and emotional disposition. Interest refers to enjoyment in mathematical activity.

Perceived competence is similarly central: drawing on Merleau-Ponty's notion of the "I can," Di Martino and Zan [16] describe it as a felt sense of capability rather than a purely cognitive judgment. Although this phenomenological perspective differs from Bandura's [5] psychological construct of self-efficacy, both approaches highlight the importance of how learners perceive their own capacity to act. Confidence, therefore, plays a crucial role in mathematical engagement. Emotional disposition arises from the interaction between cognitive appraisal and physiological responses [12, 33], highlighting the close relationship between cognition and emotion. In this study, task enjoyment is considered as an indicator of pupils' emotional disposition toward the activity, while perceived performance is related to perceived competence and self-efficacy.

During problem solving, students encounter difficulties influenced by both internal factors (knowledge, skills, beliefs) and external factors (task structure and wording). These factors influence perceived difficulty, which is not an objective property of a task but a subjective experience [40]. Often related to metacognition, perceived difficulty can be understood as a monitoring process of ongoing cognitive activity [21], influencing self-regulation, emotions, and strategy use [18]. Perceived difficulty is particularly relevant here because it captures pupils' subjective evaluation of the task, rather than an objective property of the activity. As such, it can influence persistence, self-regulation, and willingness to continue engaging with mathematical tasks.

Alongside attitudes and metacognition, values also play a central role in mathematics learning. Mathematics is culturally and socially situated rather than value-neutral [7, 38]. Different frameworks distinguish epistemological, social, cultural, and personal values [28], as well as mathematical, educational, and general educational values [7]. Values are deeply held beliefs about what is important, guiding actions and decisions [14] and they differ from beliefs, which concern what is considered true [39].

In line with this perspective, Albano et al. [1], focusing on Grade-2 pupils' interaction with one story-problem, showed that students who reported more positive attitudes toward the activity also tended to provide more elaborated answers, confirming the role of affective dimensions in mathematical problem-solving processes.

2.4. Integrating the perspectives

Taken together, these three strands suggest that addressing learning loss requires more than providing additional practice. It calls for the design of learning environments that are meaningful, engaging, and supportive of both cognitive and affective processes. In this perspective, informal learning contexts can become valuable spaces for mathematical activity if they are carefully structured to include: (i) meaningful and contextualized tasks, (ii) opportunities for explanation and reflection, and (iii) forms of mediation that support engagement without reducing autonomy.

Accordingly, learning is understood here not merely as the accumulation or retention of measurable knowledge, but as sustained participation in meaningful mathematical activity, involving reasoning, representation, argumentation, affective engagement, and adult mediation. This perspective shifts the focus from a deficit view of summer interruption to the conditions that support continuity of mathematical practices beyond the classroom.

The integration of storytelling and modelling offers a promising approach in this direction, as it combines narrative engagement with mathematical reasoning. At the same time, attention to affect and engagement ensures that students are not only able, but also willing, to participate in the learning process. This integrated framework provides the theoretical foundation for the design and analysis of learning environments, such as the MaTEs project, which aim to support meaningful mathematical activity beyond the classroom.

3. The design of the MaTEs web-app

3.1. The rational underpinning the design

The MaTEs web-app is a digital tool designed to support mathematics activities during the summer period, such as informal context, with the aim of keeping students' skills active through meaningful and contextualized problems [44, 42]. The web-app's activities are not intended as review exercises, but as tools to keep mathematical thinking active over the summer, fostering meaningful and lasting learning. In relation to the phenomenon of learning loss, this type of activity seems to be particularly relevant, as the loss of learning mainly affects reasoning, problem solving, and argumentative skills, rather than basic knowledge. The activities proposed by the web-app are therefore designed with attention to the three fundamental dimensions of learning: cognitive, metacognitive, and affective [35, 16]. The three dimensions are operationalized through the design of the tasks. The cognitive dimension is addressed by non-routine story-problems involving understanding, representation, modelling, and argumentation; the metacognitive dimension by prompts asking pupils to explain strategies and reflect on their work; and the affective dimension by narrative contexts, symbolic feedback, and self-reported evaluations of enjoyment, perceived performance, and perceived difficulty. These principles are further detailed in Sections 3.2 and 3.3.

Alongside problem solving, the web-app promotes explicit moments of reflection in which students are invited to explain the strategies they adopted, review their work, and develop awareness of their own learning process. Therefore, the MaTEs web-app is built to support pupils' engagement with non-routine tasks, encourage argumentation, and make visible the interplay between mathematical reasoning, affective dimensions, and social values [1, 3]. Thus, the design of the MaTEs web-app is not merely a technical organization of tasks, but an operational translation of the conceptual framework: informal learning is addressed through adult-mediated home activities; storytelling supports sense-making and engagement; modelling and representations structure pupils' mathematical work; and affective and metacognitive prompts make pupils' subjective experience and reflection visible.

It should be noted that the web-app is not intended as an automated learning environment, but rather as a mediating device between the child, the adult, and the mathematical activity according to the informal learning approach (see Section 2.1). The use of the web-app is not the responsibility of the child, but it is an experience mediated by a reference adult (a parent or another adult). In fact, the web-app is a tool in the hands of the adult, used both to present the problem-story and the individual activities (through text, audio, and images) and to collect the students' work. Moreover, as a distinctive feature of the web-app, the adult can and is required to upload the child's work, typically in the form of images (photos of drawings) and audio recordings, which document the thinking process and not just the final result. The role of the adult is crucial in this process, as it supports the child in interpreting the task, verbalizing their reasoning, and reflecting at the end, making it possible to use the web-app in a way that is not merely procedural, but deeply formative.

The MaTEs web-app can therefore be understood as a hybrid environment, in which digital and offline dimensions are integrated. Its functioning is based on some pivotal elements: centrality of narration, active role of the child, mediation by the adult, the valorisation of the process as well as the result and the inclusion of metacognitive reflection moments. Thus, the web-app is not a substitute for the teacher, but a device that makes it possible, even in the summer context, to offer a guided learning experience in which the child can continue to engage meaningfully with mathematics.

3.2. The structure of the web-app

The structure of the web-app reflects the design principles described above by organizing each activity as a guided but flexible sequence, in which digital prompts, adult mediation, offline mathematical work, and documentation are progressively integrated.

The web-app is organized into stories, each structured as a sequence of 6 or 7 tasks. Through the interface, the user can select a story and progressively access the related tasks. As shown in Figure 1, each task is presented with a clear structure: an initial screen with a title, image, and text, accompanied by the option to start audio playback that reads the instructions. Each task consists of a short narrative situation, a request (e.g., draw, solve a problem, explain own reasoning), suggestions for the reference adult, guiding questions to stimulate the child's reflection and a final phase of feedback and metacognition.

This structure creates a continuous alternation between digital moments (presentation, uploading, feedback) and concrete moments (paper-based work, dialogue, reflection). The web-app introduces and guides the activities, but meaning making mainly occurs in offline work. From the child's perspective, the interaction is simple and structured: they listen to or read a story, work on a task, explain what they did, and receive symbolic feedback (the sunflower grows). At the same time, the adult is involved as a facilitator, providing guidance that supports the child without directing them toward a single answer. The web-app offers a clear structure that allows the adult to know what to propose, how to accompany and support the child, and how to collect and document the work.

The interface is simple but structured: the presence of buttons (audio, photo, microphone), status messages (task completed), and visual feedback makes the experience accessible and guided, while still leaving room for the child's autonomy in the problem-solving process. The content is presented to the child through different modes: the adult may read the text, play the audio, or invite the child to read independently. This flexibility allows the activity to be adapted to the child's characteristics and the context. The task is not carried out directly within the app: after reading or listening, the device is set aside, and the work continues offline.

The central phase of the activity takes place outside the web-app. The child develops the response using paper and pencil, drawings, or oral explanations. The adult accompanies the process without replacing the child, supporting him/her in the comprehension of the text, the exploration of different strategies and the verbalization of his/her reasoning. The suggestions provided in the web-app are specifically addressed to the adult and serve to guide this medi-

Figure 1. Screenshots of the MaTEs web app interface showing (from left to right): login page, activity selection dashboard, structure of a story with sequential tasks, and an example of a task including instructions, suggestions, upload, and feedback features.

ation. Another distinctive element is the presence of a symbolic feedback system (the growth of the sunflower), which accompanies the pathway and contributes to building a narrative dimension also at a motivational level.

3.3. The story-problems: features and contents

The story-problems constitute the main design component through which the conceptual framework is enacted. Their features are directly connected to the three strands discussed above: informal learning through adult mediation, storytelling and modelling through narrative mathematical tasks, and affective engagement through meaningful contexts, openness, argumentation, and reflection.

The story-problems in the web-app share several common design features, based on the conceptual background. They allow for freedom of exploration: students can follow different paths and arrive at different solutions, moving beyond the idea of a single correct answer. This openness is closely connected to the request for argumentation: they explicitly require argumentation, shifting the focus from the final result to the reasoning process and the justification of choices made. In this process, a key element is the use of representations, particularly drawing (see Section 2.2), which is introduced from the earliest stages of the activity as a way to understand and reinterpret the story. The activities are also designed to encourage the emergence of multiple possible mathematical models, challenging the idea of mathematics as unique and absolute. Moreover, the story-problems integrate the mathematical dimension with social and ethical aspects, as the characters' decisions and proposed solutions involve values such as fairness, justice, and cooperation.

Each story is structured as a pedagogical sequence involving alternating phases of web-app mediation, adult support, and pupils' offline mathematical activity. The web-app mainly supports the presentation, guidance, and documentation of the activity, while the core mathematical work is carried out by the pupil, typically mediated by the adult. The sequence unfolds through the following phases; in parenthis, we indicate who is directily involved:

(i) *Narrative introduction*: (pupil supported by the adult within the app) presentation of characters and context through text and audio, allowing the child to build an initial understanding of the situation. This phase contributes most to the understanding of the real situation in the modelling cycle.

(ii) *Initial representation (drawing)*: (pupil supported by the adult) the child is invited to draw the story, activating a personal representation of the situation. This phase contributes most to the structuring and presenting steps in the modelling cycle.

(iii) *Exploration*: (pupil supported by the adult) initial questions guide observation, comparison, and hypothesis formulation. This phase supports mostly the transition from situational understanding to early mathematization steps.

(iv) *Structuring and modelling*: (pupil) activities require identifying relationships between quantities and developing solution strategies, corresponding mainly to the mathematizing steps.

(v) *Openness to multiple solutions*: (pupil) some tasks allow for more than one possible answer, promoting flexibility and comparison of strategies, contributing mainly to the mathematizing and working mathematically steps.

(vi) *Argumentation*: (pupil supported by the adult with the app) the child is asked to explain and justify their reasoning, often through audio recordings. This phase contributes most to the interpreting, and working mathematically steps.

(vii) *Re-elaboration*: (pupil) the introduction of new information or constraints requires revising and refining strategies, contributing mainly to interpreting, validating, and presenting steps.

(viii) *Documentation and metacognitive reflection*: (pupil supported by the adult within the app) each activity ends with the upload of the produced work (drawing and audio) and a reflection on enjoyment, perceived success, and difficulty. This phase contributes most to the interpreting presenting steps.

This structure operationalizes the mathematical modelling process [9]. Although there is not a rigid one-to-one correspondence between phases and modelling steps, the activities are designed to support iterative and intertwined transitions from reality to mathematics and back again through the steps of the modelling cycle (see section 2.2).

This design choice shows how narrative mathematical tasks can open up multiple modelling paths and make explicit the values underlying different mathematical solutions, such as equity, fairness, and cooperation [3].

It is worthwhile to highlight that, taken together, the app's stories constitute a didactic tool aimed at developing deep mathematical competencies, particularly problem solving, argumentation, and modelling, through meaningful and contextualized activities. In relation to the phenomenon of learning loss, these activities are especially relevant, since learning loss in mathematics mainly concerns these competencies rather than basic knowledge.

In the following we briefly describe the content of the six stories presented in the app:

Story 1: Grandma Adele's Candies The story revolves around the preparation of a party, where children must manage bags of candies of different flavours. It introduces situations involving choice and the distribution of objects (candies), engaging the four operations with natural numbers up to 100.

Story 2: Naima and the Parking Lot The story is based on an everyday situation: a morning of errands in the city. Here, children work on spatial and organizational aspects related to the arrangement of objects in space.

Story 3: The Farm of Animals The story takes place on a farm and presents a classic problem reinterpreted in a narrative form: the heads and legs problem. The narrative introduces pre-algebraic and arithmetic elements.

Story 4: The Kingdom of Regiomonte This story introduces a fantasy narrative setting in which a prince must manage bridges between islands in order to recover pieces of the crown. It addresses geometric and topological content through playful, experiential contexts (paths and mazes) that are familiar to students from early childhood education.

Story 5: Chocolate Bars The story starts from a familiar situation (a birthday party) and progressively introduces more complex mathematical concepts. It introduces problems related to measurement and equivalence (e.g., comparing a 4×4 chocolate bar with a 2×8 one), gradually leading to the concept of fractions as a ratio between natural numbers.

Story 6: Beremiz, the Man Who Knew How to Count This story, adapted from a narrative tale, introduces a scenario of sharing and reward. It presents a situation involving the division of resources that allows for multiple mathematical solutions.

3.4. Implementation and usage of the app

Interaction with the web-app unfolds in several phases, alternating online and offline work. The adult accesses the platform, identifies the story to propose, and selects the task, then acts as a director, deciding timing and modes of presentation. Thus, the use of the web-app is explicitly designed as an experience mediated by the reference adult, who plays a central role in all phases of the activity.

By selecting a story, one accesses a screen that presents a recurring structure: a reference image representing the situation, the task instructions, an introductory text, a button to listen to the story audio and the list of tasks that make up the pathway, a section dedicated to suggestions for the reference adult. Each task appears as a selectable element; once completed, it changes status (e.g., from orange to green), making progress visible. Once the task work is completed, the adult helps the child upload the product (drawing) and an audio comment to the app.

The web-app requires both: photo and audio. Only after both are provided the "done" button becomes active. This constraint is designed to encourage verbalization and argumentation of the work carried out.

As the final step, the children are guided along a metacognitive reflection, where they are invited to express their own point of view about the task enjoyment, the perceived performance and the perceived difficulty, by answering three questions: *Did you like it? Do you think you did well? Did it seem difficult?* using a three-point scale (not at all / somewhat / very). In addition, an audio comment is required, guided by specific prompts (e.g., about the drawing or the strategy used).

Because of the structure of the web-app, the design of the activities, and the intended mode of interaction, the term user does not refer exclusively to either the adult or the pupil, but to the adult-pupil pair. Adults are users in that they interact with the web-app from a technical perspective and facilitate pupils' access to the mathematical activities. Pupils, in turn, are users in that they engage with the mathematical content and actively construct meaning. At the same time, pupils also use the web-app to record their responses (e.g., through audio), while adults may or may not be directly involved in the mathematical work.

4. Research Question

The web-app, designed according to the conceptual background and the aim of the project, can provide fine-grained traces of students' activity, allowing the identification of different behaviour patterns over time.

These patterns are not interpreted as direct evidence of learning outcomes, nor as equating web-app use with learning. Rather, these patterns may reflect distinct trajectories of usage, which in turn can offer insights into the continuity of learning beyond the classroom. Thus, while the conceptual framework justifies the design of the learning environment, the present analysis focuses on its use, investigating whether log data reveal different ways in which users engaged with the proposed mathematical activities over the summer. Against this background, the present study addresses the following research question:

> To what extent do different behaviour patterns of the web app during summer reflect distinct trajectories of usage, and what do they reveal about continuity of engagement with mathematical activity?

5. Methodology

5.1. Participants and data collection

The participants are pupils who used the web app during the summer 2024 between the end of grade 2 and the beginning of grade 3 (September 2024). They come from 44 school institutes in two different metropolitan cities (Milan and Naples). The web app recorded 616 families, but 315 of whom completed at least one task.

The activities were released progressively over the summer period according to a predefined schedule (see Figure 2), with sequential access requiring completion of prior activities. In particular, the six stories were made available at regular intervals between early July and early September 2024. A longer interval was introduced between Story 3 and Story 4, with the latter released on August 18, reflecting an intentional pause during the mid-August holiday period. The platform remained accessible until September 23, marking the end of the observation period, one week after the start of the school year.

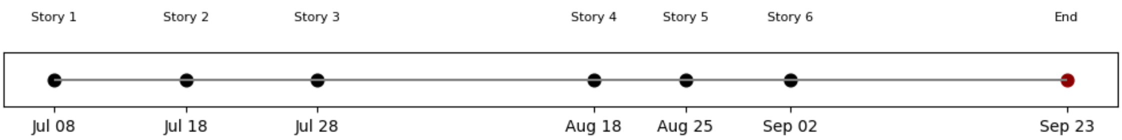

Figure 2. Timeline of the release of the six activities during the summer period. Each point represents the release date of a story, while the final marker indicates the end of the observation period.

We collect data from user logs. More specifically, for each user identified by an ID number, we obtain the following information: the story accessed, the task accessed, whether the user completed the task.

The dataset consists of anonymized log data extracted from the MaTEs web-app, where each record corresponds to a user-task interaction. For each entry, the dataset includes: a user identifier, timestamps of submission and last update, the story and task identifiers (with corresponding titles), and information on task completion: the images submitted (i.e., photos of drawings made by the child), and the audio recordings (see Section 3.3). Additional variables report the users' metacognitive reflection on the activity, collected through three evaluation items (see Section 3.4).

5.2. Methods of analysis

This study adopts a quantitative approach based on the cluster analysis of log data to identify patterns of user behaviour and statistical tests to compare different profiles of interaction with the web-app .

The preprocessing of the log data was carried out starting from the original Excel file containing users' interactions with the web-app . The dataset was first filtered by retaining only completed tasks and removing duplicate records at the user-task level, keeping the earliest completion timestamp. Timestamps were then discretized into six predefined time windows (DT, i.e., discrete time windows between two release dates), corresponding to the release schedule of the activities (see Figure 2). Based on this discretization, a user-by-time matrix was constructed, where each entry represents the number of tasks completed by a given user within each time window DT. From this representation, cumulative trajectories were computed by applying a cumulative sum across time windows, thus capturing the progressive evolution of users' activity.

In addition, a set of derived temporal variables was constructed to summarize different aspects of user behaviour. These included the total number of completed tasks (i.e., completion CT), the number of active time windows (AC), the index of the first and last active window (FW, LW), the maximum number of consecutive active windows (MC) and the proportion of activity concentrated in the final time windows (LT). These variables were selected to capture complementary dimensions of engagement that are particularly relevant in the context of informal learning during extended school breaks. Specifically, CT reflects the overall intensity of engagement, while AC provides an indication of its temporal distribution across the observation period. FW and LW allow identifying the timing of activation and disengagement, thus distinguishing early starters, late participants, or sustained users. MC captures continuity of engagement by measuring the extent to which activity is maintained across consecutive time windows. Finally, LT reflects the degree of temporal concentration of activity in the final phase, allowing the identification of "last-minute" usage patterns that resemble homework completion behaviours. Taken together, these variables provide a multidimensional and interpretable representation of users' temporal trajectories, enabling a more fine-grained comparison of engagement patterns across clusters.

To explore usage patterns of the web app in a data-driven way, a time series clustering analysis based on Time Series K-Means methodological approach [26] was conducted. For each user, a time series was constructed representing the cumulative number of completed tasks within each interval. This representation allows capturing users' trajectories of progression over time, focusing on the evolution of engagement rather than on isolated events.

Clustering was performed using the python TimeSeriesKMeans algorithm with Euclidean distance applied to the cumulative series, without prior standardization, in order to preserve information about absolute activity levels in addition to trajectory shapes. The number of clusters was determined using the Elbow method by analysing the trend of inertia as a function of k. In particular, the relative decrease in inertia between successive values of k was examined to identify a point of inflection beyond which increasing the number of clusters yields only marginal improvements in within-cluster variability. The final choice of k was therefore informed by both the behaviour of inertia and the interpretability of the resulting patterns.

To assess the statistical significance of the differences among clusters, a non-parametric approach was adopted. Given the non-normal distribution of the derived variables and the unequal group sizes, the Kruskal–Wallis test [30] was applied to compare the distributions of each variable across clusters. This test evaluates whether at least one group differs from the others in terms of median values. For each variable, the null hypothesis of equal distributions across clusters was tested independently. When the Kruskal–Wallis test indicated significant differences, post-hoc pairwise comparisons were conducted using Dunn's test with Bonferroni correction [20] to control for multiple comparisons. This procedure allowed identifying which specific pairs of clusters differed significantly for each variable of interest, providing a detailed characterization of the statistical separation between clusters. Violin plots were used to represent the variability of each cluster with respect to both the time windows and the chosen derived variables.

To investigate whether clusters differ in terms of metacognitive variables (see Section 3.4) data were extracted from the original dataset and aggregated at the user level. Since metacognitive variables were collected at the task level, average scores were computed for each user across all completed tasks, obtaining a single value per metacognitive variable and per user. More specifically, after each task, pupils answered three metacognitive feedback questions ("Did you like it?", "Do you think you did well?", "Did it seem difficult?") using a three-point response scale. Numerical scores were assigned to these responses (1 = low evaluation, 2 = intermediate evaluation, 3 = high evaluation), and the reported scores therefore represent the average level of pupils' self-reported feedback across the completed tasks. These variables, coded as a discrete scale from 1 to 3, were then merged with the cluster assignments obtained from the time series clustering procedure.

Given the ordinal nature of the data and the non-normal distribution of the scores, differences across clusters were assessed using the Kruskal-Wallis test. When appropriate, Dunn's post-hoc test with Bonferroni correction was applied to perform pairwise comparisons between clusters. In addition bar plots were used to visually inspect the distributions of feedback scores across clusters.

6. Data Analysis

As first preliminary analysis, we focused on the temporal distribution of completed tasks (see Figure 3). It shows a highly uneven pattern over time, with a moderate level of activity in the initial phase, followed by a substantial increase and a pronounced peak toward the end of the observation period. The moving average highlights a clear upward trend culminating in a concentration of activity in the final weeks, followed by a sharp decline. This overall pattern suggests that use is not uniformly distributed but rather characterized by periods of low activity interspersed with phases of intense usage.

This aggregate representation raises important questions. Although it clearly shows a global tendency toward late concentration of activity, it does not allow us to distinguish whether this pattern is shared by all users or driven by specific subgroups. In other words, the observed peak may result from heterogeneous behaviors - such as consistently active users, late starters, or users concentrating their activity at the end - that are not visible at the aggregated level. This limitation highlights the need for a more fine-grained analysis capable of identifying distinct usage trajectories, motivating the adoption of a clustering approach to profile users based on their temporal patterns of engagement.

This section unfolds as follows: first, we determine the number k of possible clusters corresponding to a specific profile using the Elbow method (Subsection 6.1), then the cluster trajectory and variability are reported (Subsection 6.2), concluding with statistical tests to characterize clusters (Subsections 6.3 and 6.4).

6.1. Determining the Number of Clusters

The inertia curve in Figure 4 shows a sharp decrease for lower values of k, followed by a more gradual reduction, with a visible change in slope around $k = 4$. The analysis of the inertia values is further supported by the examination of the relative reduction between consecutive values of k. The decrease in inertia remains high up to $k = 4$ (26.09% from $k = 2$ to $k = 3$ and 26.36% from $k = 3$ to $k = 4$), while a marked drop is observed from $k = 4$ to $k = 5$ (14.11%).

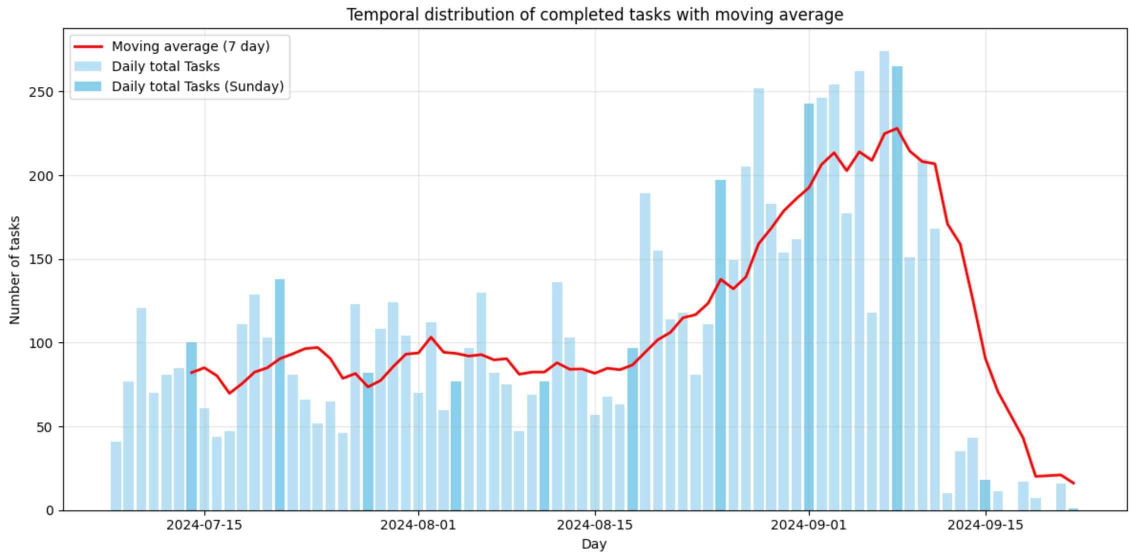

Figure 3. Daily number of completed tasks over time (bars) with a 7-day moving average (red line). The distribution highlights a non-uniform pattern of activity, with a marked increase and concentration toward the final phase of the observation period.

For higher values of k, the relative improvements progressively decrease, stabilizing around lower values. This pattern indicates that most of the structural variability in the data is captured by $k = 4$, while additional clusters account for increasingly marginal refinements. On this basis, the range $k = 4 - 6$ was selected for further investigation.

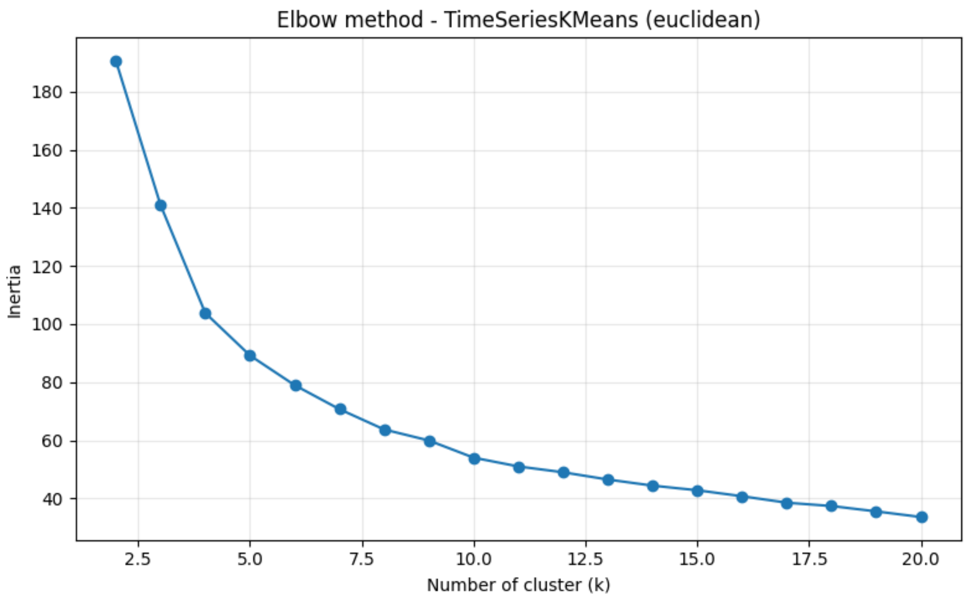

Figure 4. Inertia as a function of the number of clusters (k).

To that end, we employed cluster analysis with the three different values of k. While all configurations showed statistically significant differences across clusters with respect to the behavioural variables, as confirmed by the Kruskal-Wallis tests, the post-hoc comparisons revealed important differences in terms of separability. In particular, increasing the number of clusters led to a reduction in statistical distinctiveness, with several cluster pairs becoming indistinguishable across multiple variables for $k \geq 5$. This indicates that higher values of k introduce partially overlapping or redundant groups.

This interpretation is further supported by the proportion of non-significant pairwise comparisons observed in the Dunn post-hoc tests (Table 1), that is, comparisons between all possible pairs of clusters for each derived variable. The proportion increases from 22.2% for $k = 4$ to 30.0% for $k = 5$ and 34.4% for $k = 6$. This trend indicates a progressive reduction in cluster separability as k increases.

k	**Non-significant pairs**	**Total pairs**	**Percentage**
4	8	36	22.2%
5	18	60	30.0%
6	31	90	34.4%

Table 1. Percentage of non-significant pairwise comparisons (Dunn test) across clustering solutions, showing reduced separability as k increases.

Moreover, increasing k does not provide additional explanatory power with respect to metacognitive variables, which remain statistically homogeneous across clusters for all configurations. Taken together, these elements support the selection of $k = 4$ as the most appropriate solution, as it achieves a balance between statistical separability, interpretability, and parsimony.

6.2. Cluster Trajectories and Variability

The centroids obtained from the Time Series K-Means clustering with $k = 4$ show four distinct trajectories in terms of cumulative task completion over the considered time windows (DT1–DT6). Each centroid represents the average profile of the users assigned to the corresponding cluster. The size of the clusters are reported in Table 2.

Cluster	0	1	2	3
n. users	41	86	58	130
% of users	13.0%	27.3%	18.4%	41.3%

Table 2. Distribution of users across the four clusters.

All clusters exhibit a monotonic increase, as expected from the cumulative nature of the data, but differ in both slope and curvature (see Figure 5). Cluster 0 presents a different pattern, with low values in the initial intervals and a sharp increase in the final time windows, resulting in a pronounced change in slope between DT4 and DT6 time window. Cluster 1 is characterized by the lowest values throughout the entire time span, with a gradual and limited increase. Cluster 2 also shows a strong increase, particularly in the earlier intervals, followed by a more moderate growth in later stages. Cluster 3 displays the highest values across all time windows, with a relatively steep and a consistent growth.

Overall, the centroids highlight differences in magnitude, rate of increase, and temporal distribution of cumulative activity, indicating that the clustering procedure successfully identifies distinct trajectories within the dataset.

Figure 6 reports the violin plots which represent the distribution of cumulative task completion across time windows (DT1–DT6) for each cluster, together with the corresponding centroid trajectory. For all clusters, the distributions shift progressively toward higher values across time, reflecting the cumulative nature of the variable.

Cluster 0 exhibits a markedly different pattern, with very low values and narrow distributions in the initial windows, followed by a substantial increase in both central tendency and dispersion in the final windows (DT5–DT6). Cluster 1 is characterized by low cumulative values across all time windows, with distributions concentrated near the lower range and moderate spread. Cluster 2 also presents relatively high values, with wider distributions especially

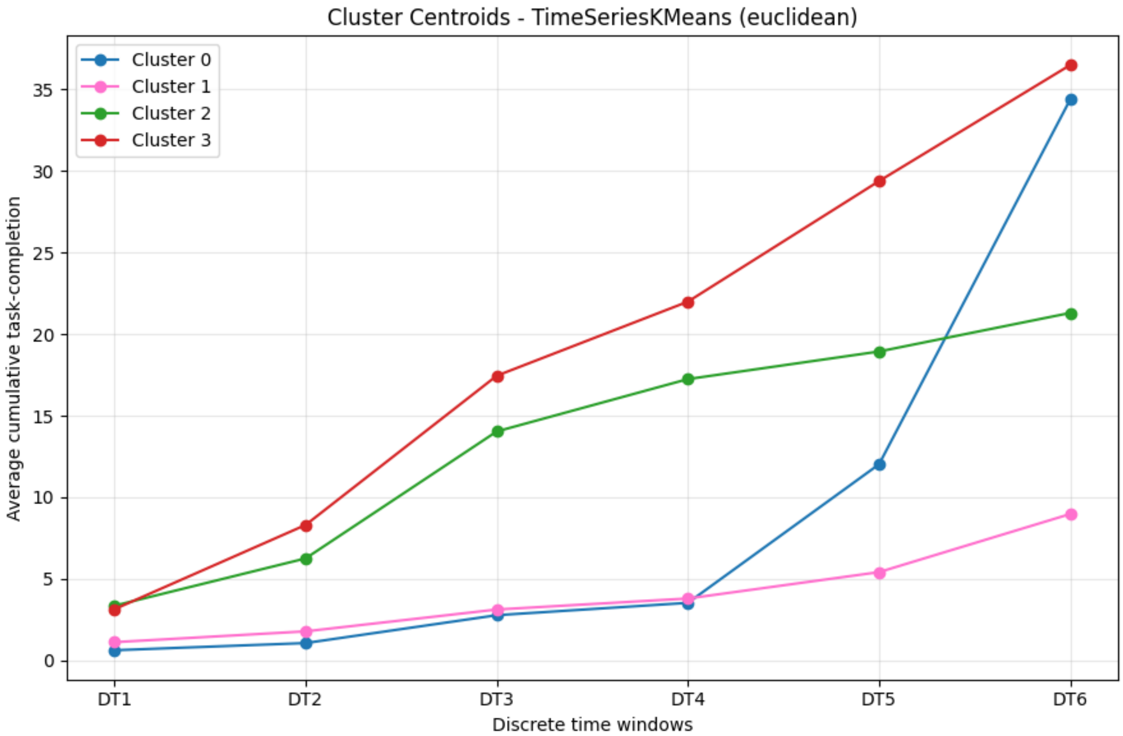

Figure 5. Centroids of the four clusters obtained with Time Series K-Means (Euclidean distance) based on cumulative task completion. The vertical axis represents the average of the cumulative number of completed tasks of each cluster, while the horizontal axis represents the discrete time windows (DT1–DT6). Each line represents the average trajectory of a cluster, obtained by averaging their cumulative task-completion time series.

in the intermediate windows (DT2–DT4), suggesting greater variability in the trajectories. Cluster 3 shows consistently higher central tendencies across all time windows, with relatively concentrated distributions around the centroid, indicating lower dispersion. Across clusters, differences emerge not only in the magnitude of cumulative values but also in the spread of the distributions, with some clusters displaying more compact shapes and others showing broader variability, particularly in later time windows.

6.3. Statistical Validation of Cluster Differences

The statistical validation of the clusters was carried out using the Kruskal-Wallis test on a set of derived variables capturing different dimensions of users' engagement (overall completion, temporal distribution, continuity, and concentration of activity). The results show that all variables (CT, AC, FW, LW, MC, LT) defined in Section 5.2 significantly differ across clusters with very low p-value ($p \ll 0.001$) indicating that the identified clusters correspond to statistically distinct patterns of usage.

The distribution of temporal variables across clusters is shown in Figure 7, using violin plots to represent both central tendency and variability within each group. Each subplot corresponds to a different variable, allowing comparison of how clusters differ across multiple dimensions of activity. The distributions reveal clear differences across clusters for several variables. In particular, completion (CT) shows strong separation, with clusters characterized by distinct ranges of cumulative activity. Similarly, AW (active_windows) and MC (max_consecutive) highlight differences in the continuity and spread of activity over time, with some clusters exhibiting more concentrated and others more distributed patterns. The variables FW (first_active) and LW (last_active) capture differences in the timing of engagement, with clusters showing distinct

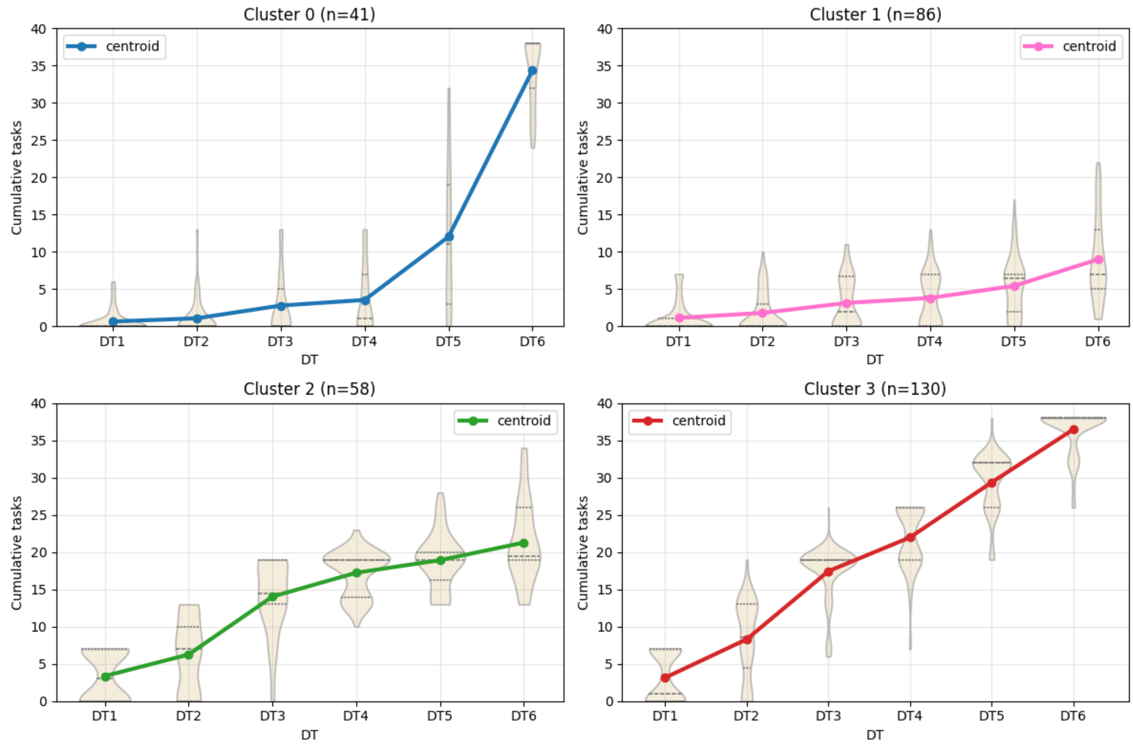

Figure 6. Distribution of cumulative tasks across time windows for each cluster ($k = 4$). Violin plots represent the variability of users' trajectories within each cluster, while the overlaid line indicates the cluster centroid.

positions along the temporal axis. Finally, LT (share_last2) exhibits a highly skewed distribution in some clusters, indicating strong concentration of activity in the final time windows for specific groups. Overall, the violin plots confirm that clusters differ not only in terms of overall magnitude of activity but also in timing and continuity, with varying degrees of dispersion across clusters depending on the variable considered.

Post-hoc comparisons using Dunn's test with Bonferroni correction confirm the structure of these differences. Most pairwise comparisons are statistically significant, especially for CT and LT, suggesting a strong differentiation between clusters in terms of overall engagement and temporal concentration of activity. At the same time, some non-significant comparisons emerge for specific variables (e.g., CT between clusters 0 and 3, or FW between clusters 0 and 1), indicating partial overlaps between certain groups. Table 3 summarizes the pairwise differences between clusters, reporting the variables for which statistically significant differences are observed. Overall, these results support the robustness of the clustering solution, while also suggesting the presence of both clearly distinct profiles and more nuanced intermediate behaviours.

	C0	**C1**	**C2**
C1	CT, AC, LW, MC, LT	–	–
C2	CT, FW, LW, LT	CT, AC, FW, MC, LT	–
C3	FW, MC, LT	CT, AC, FW, LW, MC	CT, AC, LW, MC, LT

Table 3. Pairwise comparison of clusters (C0, C1, C2, C3) based on Dunn post-hoc tests (Bonferroni corrected). Each cell reports the variables for which the two clusters differ significantly ($p << .001$). Abbreviations: completion (CT), active windows (AC), first active window (FW), last active window (LW), maximum consecutive activity (MC).

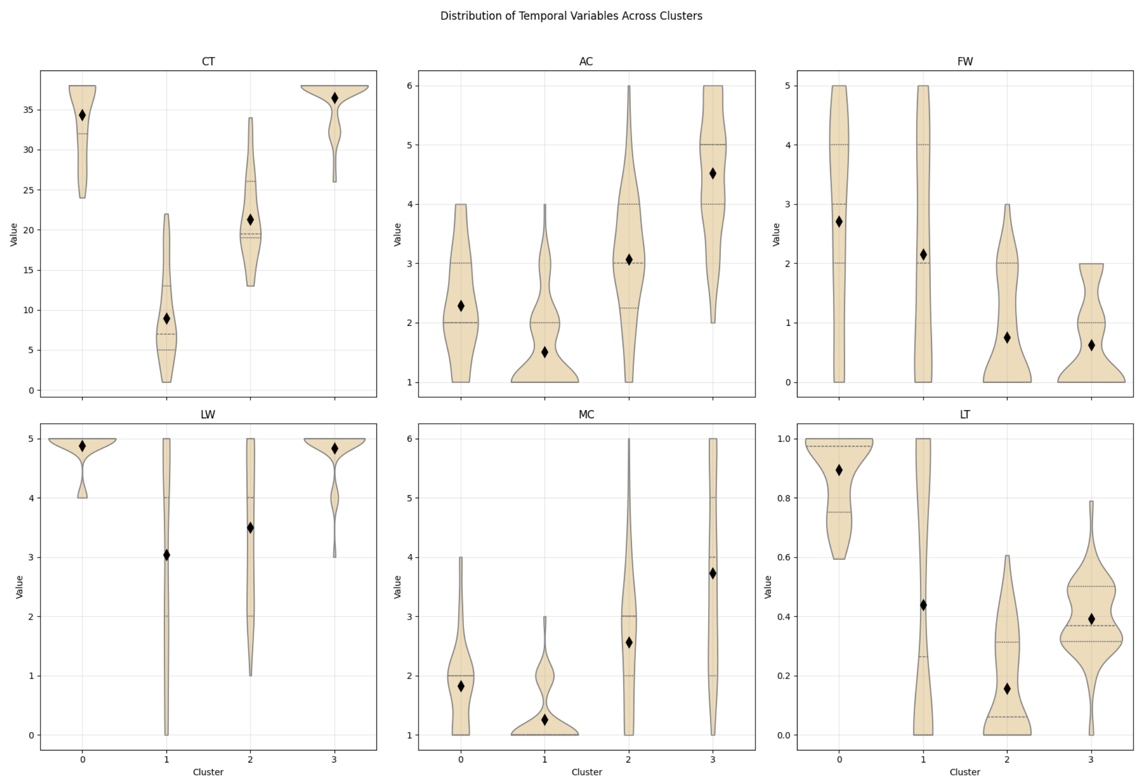

Figure 7. Distribution of temporal variables across clusters. Violin plots show the variability within each cluster for completion (CT), active windows (AC), first active window (FW), last active window (LW), maximum consecutive activity (MC), and share of activity in the final time windows (LT). Differences across clusters are observed in magnitude, timing, and continuity of activity.

6.4. Metacognition Distribution Across Clusters

Figure 8 shows the normalized distribution of metacognitive reflection scores across clusters for the three variables. For each cluster, the proportions of scores (1, 2, and 3) are displayed, allowing comparison independent of cluster size.

Across all three metacognitive variables, the distributions are highly similar between clusters. In each case, score 3 represents the largest proportion of responses, followed by score 2, while score 1 appears only marginally. The shapes of the distributions are largely overlapping across clusters, with only minor variations in the relative proportions of scores.

For task enjoyment, all clusters show a predominance of score 3, with moderate presence of score 2 and very limited occurrence of score 1. For perceived performance, the concentration on score 3 is even stronger, particularly for Cluster 2 and Cluster 3, while score 2 remains secondary and score 1 negligible.

For perceived difficulty, the pattern remains consistent, with score 3 dominating across all clusters and only slight differences in the proportion of score 2.

However, the statistical analysis of metacognitive variables did not reveal significant differences across clusters. The Kruskal-Wallis test showed non-significant results for all three feedback variables (p-value $> .05$), indicating that the distributions of user metacognitive reflections are comparable across clusters. Consistently, Dunn's post-hoc tests did not identify any significant pairwise differences between clusters. These findings suggest that, despite the clear differentiation in temporal usage patterns identified through clustering, users' metacognitive reflections of the activities remain relatively homogeneous. In other words, distinct engagement trajectories do not correspond to significantly different perceived experiences, highlighting a decoupling between patterns of use and subjective experience.

Overall, the normalized distributions confirm a high degree of similarity across clusters and a general skewness toward higher feedback values.

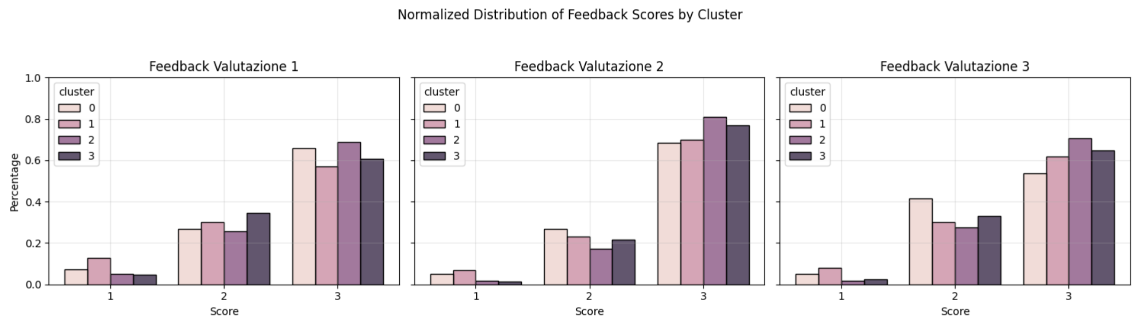

Figure 8. Normalized distribution of feedback scores across clusters. Bars represent the proportion of responses within each cluster, allowing comparison independent of cluster size.

7. Discussion and conclusion

In the previous sections we report the clustering analysis which allows us to identify four distinct usage profiles characterized by different levels of engagement, temporal distribution, and continuity of activity. This section aims at addressing the research question: "To what extent do different behaviour patterns of the web app during summer reflect distinct trajectories of usage, and what do they reveal about continuity of engagement with mathematical activity?". Accordingly, the profiles are interpreted not as direct indicators of learning outcomes, but as traces of how adult-pupil pairs organized participation in mathematical activities over the summer period. To that end, we identify profiles characterized by specific behaviour patterns and interpret them from the didactical perspective, as reported in Table 4.

Profiles [Cluster] (numerosity)	**Description**	**Didactical interpretation**
Late sprinter [0] (41)	High completion achieved through strong concentration of activity in the final time windows	Usage concentrated at the end of the summer break (homework-like pattern)
Low engagement [1] (86)	Low completion, limited activity across time windows, low continuity	Limited engagement with the activities
Early fader [2] (58)	Good initial engagement followed by a reduction of activity in later phases	Early activation not sustained over time
Ideal [3] (130)	High completion, activity distributed across time windows, high continuity	Regular and sustained use

Table 4. Characterization of the four clusters identified through time series clustering ($k = 4$), based on completion, temporal distribution, and continuity of activity.

The profiles arisen from the data analysis provide some insights and sometimes allow to substantiate some common behaviour. It is worth noting that the largest cluster is the Ideal profile, including 130 users (about 40% of the sample). At the same time, about 60% of users fall into other profiles, indicating that regular and sustained engagement was the most represented single trajectory, but not the dominant behaviour overall.

Looking at Late Sprinter, we can note that it is characterized by low initial activity followed by a sharp increase in the final phase. This profile is consistent with the well-known tendency to concentrate on school-related tasks at the end of the summer period (e.g., "homework completed at the last minute"). This allows us to argue this behaviour is not anecdotal but clearly identifiable as a distinct trajectory.

Given that the activities in the web-app are designed to foster reasoning, problem solving, and reflection rather than procedural rehearsal, comparing Late Sprinter profile with the Ideal one shows that both exhibit a high level of task completion; however, the former works intensively toward the end, while the latter works continuously. It is plausible that effectiveness of the MaTEs activities depends on continuity rather than intensity alone. However, this hypothesis cannot be directly tested within the current dataset and should be addressed in future studies.

The profile Early Fader shows initial participation followed by a progressive decline. This suggests that initial activation does not necessarily translate into sustained engagement over time, although the data do not allow identifying the causes of this decline, requiring further investigations.

The profile Low Engagement characterized by very low and sporadic activity indicates that a portion of users engages only marginally with the app. In the context of the MaTEs project, the web app is designed as a mediated environment involving both children and adults. Therefore, low engagement may reflect not only individual factors but also contextual and relational conditions of use. However, unfortunately the current data do not allow disentangling these aspects and should be considered in future research.

A particularly relevant result concerns the relationship between behavioural engagement and users' metacognitive reflection. Despite the clear differentiation in usage patterns, no significant differences emerge across clusters in terms of metacognitive variables. Recalling that the user is the pair adult-pupil, this suggests that discontinuity or late concentration of use cannot be straightforwardly attributed to lower appreciation of the app or to a more negative perceived experience. In other words, different trajectories of participation correspond to similar perceived experiences.

Overall, this study shows that interaction with a digital learning environment during the summer is a complex and heterogeneous phenomenon. The identified profiles do not simply describe different levels of use but reflect distinct ways in which adult–pupil pairs access, organize and sustain mathematical activity over time. Rather than providing direct evidence about the reduction of learning loss or improvement in achievement, the findings highlight the role that the web app may play in fostering continuity of engagement with mathematical activities during the summer period. In particular, the web app appears to support the construction of routines of participation and shared adult-pupil activities around mathematics, even in informal contexts outside school. From this perspective, the relevance of the intervention lies not only in the proposed tasks themselves, but also in the possibility of sustaining continuity of mathematical practices beyond the classroom. The collected data leads us towards further and deepening investigations as mentioned above, raising further questions such as: To what extent does late and concentrated engagement support meaningful mathematical activity, compared to distributed engagement over time? What factors influence the sustainability of engagement in informal learning environments over extended periods? Addressing such questions required complementing this quantitative analysis with a qualitative analysis of the pupils' protocols and adult mediation practices, in order to investigate whether discontinuity is related to family routines, contextual constraints, or other conditions of use.

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