Generalizations of unitarily invariant norm inequalities for matrices

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Abstract. In this article, we begin by deriving a unitarily invariant norm inequality for matrices, which is a generalization of the result due to Cao and Wu. Additionally, we introduce a matrix Cauchy-Schwarz inequality for unitarily invariant norms, further generalizing the inequality proposed by Hu.

Keywords: positive semidefinite matrix, convex function, unitarily invariant norm, Cauchy-Schwarz inequality.

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1. Introduction

Throughout this paper, let M_n denote the space of $n \times n$ complex matrices. A matrix norm $||\cdot||$ is called unitarily invariant norm if $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. Among well-known unitarily invariant norm is the Schatten p-norm, denoted by $||\cdot||_p$ and defined as $\|A\|_p = \left(\sum_{j=1}^n s_j^p(A)\right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}}, 1 \leq p < \infty$, where $s_j(A)(j = 1, 2, \cdots, n)$ are the singular values of A with $s_1 \geq s_2 \geq \cdots \geq s_{n-1} \geq s_n \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$, arranged in decreasing order. The Schatten p-norm for the values $p = 1, p = 2$ and $p = \infty$ represent the

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trace norm, the Hilbert-Schmidt norm or Frobenius norm (sometimes written as $\|A\|_F$ for that reason) and the spectral norm, respectively. Another unitarily invariant norm is the Ky Fan k-norm, denoted by $||\cdot||_{(k)}$ and defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \cdots, n$.

Kaur and Singh [1] proved that for $A, B, X \in M_n$, if A and B are positive definite, then for any unitarily invariant norm

$$\frac{1}{2}\|A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}\| \leq \left\|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}} + \alpha\left(\frac{AX+XB}{2}\right)\right\|, \tag{1.1}$$

where $\frac{1}{4} \leq \nu \leq \frac{3}{4}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Substituting A, B with A^2, B^2 and taking $u = 2\nu$ in inequality (1.1), we have

$$\frac{1}{2}\|A^{u}XB^{2-u} + A^{2-u}XB^{u}\| \leq \left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\|, \tag{1.2}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\varphi(u) = \|A^{u}XB^{2-u} + A^{2-u}XB^{u}\|$$

is convex on $[\frac{1}{2}, \frac{3}{2}]$ and attains its minimum at $u = 1$. Consequently, it is decreasing on $[\frac{1}{2}, 1]$ and increasing on $[1, \frac{3}{2}]$(See[2]). Using the convexity of the function $\varphi(u)$, Cao and Wu [3] obtained an improved version of inequality (1.2) as follows

$$\begin{aligned} &\|A^{u}XB^{2-u} + A^{2-u}XB^{u}\| \\ &\leq 2(4r_0-1)\|AXB| + 2(1-2r_0)\|A^{\frac{1}{2}}XB^{\frac{3}{2}} + A^{\frac{3}{2}}XB^{\frac{1}{2}}\| \\ &\leq 2(4r_0-1)\|AXB\| \\ &+4(1-2r_0)\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\|, \end{aligned} \tag{1.3}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$.

Let $A, B \in M_n$ and $r > 0$. Horn and Mathias proved in [4, 5] that

$$\||A^*B|^r\|^2 \leq \|(AA^*)^r\| \cdot \|(BB^*)^r\|, \tag{1.4}$$

which is a matrix Cauchy-Schwarz inequality for unitarily invariant norms.

Bhatia and Davis [6] (See also [2, p.267, Theorem IX.5.2]) got a stronger version of inequality (1.4) as follows

$$\||A^*XB|^r\|^2 \leq \||AA^*X|^r\| \cdot \||XBB^*|^r\|, \tag{1.5}$$

for $A, B, X \in M_n$ and $r > 0$, (1.5) is equivalent to

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \leq \||AX|^r\| \cdot \||XB|^r\|, \tag{1.6}$$

for positive semidefinite matrices A, B and arbitrary $X \in M_n$.

For $A, B, X \in M_n$ and A, B are positive semidefinite. Then, for every unitarily invariant norm and $r > 0$, the function

$$\psi(\nu) = \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\|$$

is convex on $[0, 1]$ and attains its minimum at $\nu = \frac{1}{2}$. Consequently, it is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$ (See [7]). Using the convexity of the function $\psi(\nu)$, Hiai and Zhan [7] gave a refinement of the inequality (1.6) as follows

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \leq \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\| \leq \||AX|^r\| \cdot \||XB|^r\|. \tag{1.7}$$

Hu [8] utilized the convexity of the function $\psi(\nu)$ to obtain an improvement of the second inequality in (1.7)

$$\begin{aligned} &\||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\| \\ &\leq 2t_0\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 + (1 - 2t_0)\||AX|^r\| \cdot \||XB|^r\|, \end{aligned} \tag{1.8}$$

where $0 \leq \nu \leq 1$, $t_0 = \min\{\nu, 1 - \nu\}$.

The unitarily invariant norm inequalities are widely applied in fields such as quantum mechanics, signal processing, data analysis and optimization theory. For example, in quantum entanglement measures, the unitarily invariant norm inequalities are employed to ensure the consistency of entanglement properties across different reference frames. Therefore, studying the unitarily invariant norm inequalities is of significant theoretical and practical importance. Many authors discussed different proofs, equivalent statements, generalizations, refinements and applications of inequalities for unitarily invariant norms. For more information on this topic, the reader is referred to [9-12] and the references therein.

This note, building on the preceding discussions, focuses on generalizing unitarily invariant norms inequalities. The structure of the note is as follows. In Section 2, we generalize inequalities (1.3) and (1.8) by using the convexity of functions. Finally, Section 3 provides concluding remarks.

2. Main results

We begin this section with the following lemma, which is useful in the proof of our results.

Lemma 2.1([13]). *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2). Then for $x_1 \le x \le x_2$, we have*

$$f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2-x_1}.$$

Theorem 2.1. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite. Then, for any unitarily invariant norm*

$$\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \le \frac{4(\mu_0-\nu_0)}{2\mu_0-1}\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| + \frac{2\nu_0-1}{2\mu_0-1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|,$$

where $\frac{1}{2} \le \nu \le \frac{3}{2}$, $\frac{1}{2} < \mu < \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$, $\nu_0 = \min\{\nu, 2-\nu\}$ and $\mu_0 = \min\{\mu, 2-\mu\}$.

Proof. For $\frac{1}{2} \le \nu \le \mu \le 1$, by the convexity of function $\varphi(\nu) = \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\|$ and Lemma 2.1, we get

$$\varphi(\nu) \le \frac{\varphi(\mu)-\varphi(\frac{1}{2})}{\mu-\frac{1}{2}}\nu - \frac{\frac{1}{2}\varphi(\mu) - \mu\varphi(\frac{1}{2})}{\mu-\frac{1}{2}},$$

which is equivalent to

$$\varphi(\nu) \le \frac{2\nu-1}{2\mu-1}\varphi(\mu) + \frac{2(\mu-\nu)}{2\mu-1}\varphi\left(\frac{1}{2}\right), \tag{2.1}$$

combining (1.2) with (2.1), we have

$$\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \le \frac{4(\mu-\nu)}{2\mu-1}\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| + \frac{2\nu-1}{2\mu-1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|$$

which is equivalent to

$$\begin{aligned} &\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \\ &\le \frac{4(\mu_0-\nu_0)}{2\mu_0-1}\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+\frac{2\nu_0-1}{2\mu_0-1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|. \end{aligned} \tag{2.2}$$

For $1 < \mu \le \nu \le \frac{3}{2}$, by Lemma 2.1, it follows that

$$\varphi(\nu) \le \frac{\varphi(\frac{3}{2})-\varphi(\mu)}{\frac{3}{2}-\mu}\nu - \frac{\mu\varphi(\frac{3}{2}) - \frac{3}{2}\varphi(\mu)}{\frac{3}{2}-\mu},$$

which implies

$$\varphi(\nu) \le \frac{2\nu-2\mu}{3-2\mu}\varphi\left(\frac{3}{2}\right) + \frac{3-2\nu}{3-2\mu}\varphi(\mu), \tag{2.3}$$

combining (1.2) with (2.3), we have

$$\begin{aligned}\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \le \frac{4(\nu-\mu)}{3-2\mu}&\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+ \frac{3-2\nu}{3-2\mu}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \\ &\le \frac{4(\mu_0-\nu_0)}{2\mu_0-1}\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+\frac{2\nu_0-1}{2\mu_0-1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|.\end{aligned} \tag{2.4}$$

It follows from (2.2),(2.4) and $\frac{1}{2} \le \nu \le \frac{3}{2}$, $\frac{1}{2} < \mu < \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$, $\nu_0 = \min\{\nu, 2-\nu\}, \mu_0 = \min\{\mu, 2-\mu\}$ that

$$\begin{aligned}\|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\| \le \frac{4(\mu_0-\nu_0)}{2\mu_0-1}&\left\|(1-\alpha)AXB + \alpha\left(\frac{A^2X+XB^2}{2}\right)\right\| \\ &+ \frac{2\nu_0-1}{2\mu_0-1}\|A^\mu XB^{2-\mu} + A^{2-\mu}XB^\mu\|.\end{aligned}$$

This completes the proof. □

Remark 2.1. Let $\mu = 1$ in Theorem 2.1, we obtain the inequality (1.3).

Remark 2.2. When $\frac{1}{2} \le \nu \le \mu \le 1$ or $1 < \mu \le \nu \le \frac{3}{2}$, the inequality in Theorem 2.1 is better than inequality (1.3).

By the convexity of function $\varphi(\nu) = \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\|$ and Lemma 2.1, we know that inequality (1.3) is equivalent to

$$\varphi(\nu) \le 2(1-\nu)\varphi\left(\frac{1}{2}\right) + (2\nu-1)\varphi(1),\ \frac{1}{2} \le \nu \le 1$$

and

$$\varphi(\nu) \le (3-2\nu)\varphi(1) + 2(\nu-1)\varphi\left(\frac{3}{2}\right),\ 1 < \nu \le \frac{3}{2}.$$

For $\frac{1}{2} \le \nu \le \mu \le 1$, since $\varphi(\nu) = \|A^\nu XB^{2-\nu} + A^{2-\nu}XB^\nu\|$ is convex on $[\frac{1}{2}, 1]$, it follows by a slope argument that

$$\frac{\varphi(\mu) - \varphi(\frac{1}{2})}{\mu - \frac{1}{2}} \le \frac{\varphi(1) - \varphi(\frac{1}{2})}{1 - \frac{1}{2}}.$$

By a small calculation, we have

$$2(1-\nu)\varphi\left(\frac{1}{2}\right)+(2\nu-1)\varphi(1)-\left[\frac{2(\nu-\mu)}{2\mu-1}\varphi\left(\frac{1}{2}\right)+\frac{2\nu-1}{2\mu-1}\varphi(\mu)\right]$$
$$=\frac{2\nu-1}{2}\left[\frac{\varphi(1)-\varphi\left(\frac{1}{2}\right)}{\frac{1}{2}}-\frac{\varphi(\mu)-\varphi\left(\frac{1}{2}\right)}{\mu-\frac{1}{2}}\right]$$
$$\geq 0.$$

For $1<\mu\leq\nu\leq\frac{3}{2}$, since $\varphi(\nu)=\|A^{\nu}XB^{2-\nu}+A^{2-\nu}XB^{\nu}\|$ is convex on $[1,\frac{3}{2}]$ and $\varphi(\nu)$ is increasing on $[1,\frac{3}{2}]$, we have

$$0\leq\frac{\varphi(\frac{3}{2})-\varphi(1)}{\frac{1}{2}}\leq\frac{\varphi(\mu)-\varphi(\frac{3}{2})}{\mu-\frac{3}{2}}.$$

By a small calculation, we have

$$(3-2\nu)\varphi(1)+2(\nu-1)\varphi\left(\frac{3}{2}\right)-\left[\left(\frac{2\nu-2\mu}{3-2\mu}\right)\varphi\left(\frac{3}{2}\right)+\frac{3-2\nu}{3-2\mu}\varphi(\mu)\right]$$
$$=(\nu-1)\frac{\varphi(\frac{3}{2})-\varphi(1)}{\frac{1}{2}}+\frac{3-2\nu}{2}\frac{\varphi(\mu)-\varphi(\frac{3}{2})}{\mu-\frac{3}{2}}$$
$$\geq 0.$$

Obviously, Theorem 2.1 is a generalization of inequality (1.3).

In the following, we utilize the convexity of the function $\psi(\nu)=|||A^{\nu}XB^{1-\nu}|^r\|\cdot|||A^{1-\nu}XB^{\nu}|^r\|$ to obtain a matrix Cauchy-Schwarz inequality for unitarily invariant norms that leads to a generalization of inequality (1.8).

Theorem 2.2. *Let $A,X,B\in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm*

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\|\cdot|||A^{1-\nu}XB^{\nu}|^r\|\\&\leq(1-r_0)|||AX|^r\|\cdot|||XB|^r\|+r_0|||A^{\mu}XB^{1-\mu}|^r\|\cdot|||A^{1-\mu}XB^{\mu}|^r\|,\end{aligned}\tag{2.5}$$

where $r>0$, $0\leq\nu\leq1$, $0<\mu<1$ and $r_0=\begin{cases}\frac{\nu}{\mu}, & 0\leq\nu\leq\mu,\\ \frac{1-\nu}{1-\mu}, & \mu<\nu\leq1.\end{cases}$

Proof. Inequality (2.5) is obvious for $\nu=0,\ \mu,\ 1$. For $0<\nu<\mu$, since $\psi(\nu)=|||A^{\nu}XB^{1-\nu}|^r\|\cdot|||A^{1-\nu}XB^{\nu}|^r\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(\nu)-\psi(0)}{\nu-0}\leq\frac{\psi(\mu)-\psi(0)}{\mu-0},$$

then

$$\psi(\nu)\leq\left(1-\frac{\nu}{\mu}\right)\psi(0)+\frac{\nu}{\mu}\psi(\mu).$$

Therefore,

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq \left(1-\frac{\nu}{\mu}\right)|||AX|^r\| \cdot |||XB|^r\| + \frac{\nu}{\mu}|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|.\end{aligned} \tag{2.6}$$

For $\mu < \nu < 1$, since $\psi(\nu)$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(\nu)-\psi(\mu)}{\nu-\mu} \leq \frac{\psi(1)-\psi(\mu)}{1-\mu},$$

then

$$\psi(\nu) \leq \left(1-\frac{1-\nu}{1-\mu}\right)\psi(1) + \frac{1-\nu}{1-\mu}\psi(\mu).$$

Therefore,

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq \left(1-\frac{1-\nu}{1-\mu}\right)|||AX|^r\| \cdot |||XB|^r\| + \frac{1-\nu}{1-\mu}|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|.\end{aligned} \tag{2.7}$$

It follows from (2.6), (2.7) and $r > 0$, $0 \leq \nu \leq 1$, $0 < \mu < 1$,

$$r_0 = \begin{cases} \dfrac{\nu}{\mu}, & 0 \leq \nu \leq \mu, \\ \dfrac{1-\nu}{1-\mu}, & \mu < \nu \leq 1 \end{cases}$$

that

$$\begin{aligned}&|||A^{\nu}XB^{1-\nu}|^r\| \cdot |||A^{1-\nu}XB^{\nu}|^r\| \\ &\leq (1-r_0)|||AX|^r\| \cdot |||XB|^r\| + r_0|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|.\end{aligned}$$

The proof is completed. □

Remark 2.3. Let $\mu = \frac{1}{2}$ in Theorem 2.2, we obtain the inequality (1.8).

Remark 2.4. When $0 \le \nu \le \mu \le \frac{1}{2}$ or $\frac{1}{2} < \mu \le \nu \le 1$, the inequality in Theorem 2.2 is better than inequality (1.8).

By the convexity of function $\psi(\nu) = \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\|$ and Lemma 2.1, we know that inequality (1.8) is equivalent to

$$\psi(\nu) \le (1-2\nu)\psi(0) + 2\nu\psi\left(\frac{1}{2}\right),\ 0 \le \nu \le \frac{1}{2} \tag{2.8}$$

and

$$\psi(\nu) \le (2\nu-1)\psi(1) + 2(1-\nu)\psi\left(\frac{1}{2}\right),\ \frac{1}{2} \le \nu \le 1. \tag{2.9}$$

For $0 \le \nu \le \mu \le \frac{1}{2}$, compared with inequality (2.8)

$$\begin{aligned}&(1-2\nu)\psi(0)+2\nu\psi\left(\frac{1}{2}\right)-\left[\left(1-\frac{\nu}{\mu}\right)\psi(0)+\frac{\nu}{\mu}\psi(\mu)\right]\\ =&\nu(2(\psi\left(\frac{1}{2}\right)-\psi(0))-\frac{\nu}{\mu}(\psi(\mu)-\psi(0))).\end{aligned}$$

Since $\psi(\nu) = \||A^\nu XB^{1-\nu}|^r\| \cdot \||A^{1-\nu}XB^\nu|^r\|$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi\left(\frac{1}{2}\right)-\psi(0)}{\frac{1}{2}-0} \ge \frac{\psi(\mu)-\psi(0)}{\mu-0},$$

that is

$$2(\psi\left(\frac{1}{2}\right)-\psi(0)) - \frac{1}{\mu}(\psi(\mu)-\psi(0)) \ge 0,$$

thus, we have

$$(1-2\nu)\psi(0) + 2\nu\psi\left(\frac{1}{2}\right) \ge \left(1-\frac{\nu}{\mu}\right)\psi(0) + \frac{\nu}{\mu}\psi(\mu).$$

For $\frac{1}{2} < \mu \le \nu \le 1$, compared with inequality (2.9)

$$\begin{aligned}&(2\nu-1)\psi(1) + 2(1-\nu)\psi\left(\frac{1}{2}\right) - \left[\left(1-\frac{1-\nu}{1-\mu}\right)\psi(1) + \frac{1-\nu}{1-\mu}\psi(\mu)\right]\\ &\qquad = (1-\nu)\left(\frac{\psi(1)-\psi(\mu)}{1-\mu} - \frac{\psi(1)-\psi(\frac{1}{2})}{\frac{1}{2}}\right).\end{aligned}$$

Since $\psi(\nu)$ is convex on $[0,1]$, it follows by a slope argument that

$$\frac{\psi(1)-\psi(\mu)}{1-\mu} \ge \frac{\psi(1)-\psi(\frac{1}{2})}{\frac{1}{2}}.$$

Thus, we have

$$(2\nu-1)\psi(1) + 2(1-\nu)\psi\left(\frac{1}{2}\right) \ge \left(1-\frac{1-\nu}{1-\mu}\right)\psi(1) + \frac{1-\nu}{1-\mu}\psi(\mu).$$

Obviously, Theorem 2.2 is a generalization of inequality (1.8).

3. Conclusion

In recent years, there has been a growing interest in exploring unitarily invariant norms inequalities. By utilizing the convexity of the functions $\varphi(\nu)$ and $\psi(\nu)$, we introduce two new matrix inequalities for unitarily invariant norms, which generalize several previously known results. The inequalities derived in this work lead to refinements of unitarily invariant norms inequalities under specific conditions. Future research will further explore these topics.

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References

[1] R. Kaur, M. Singh, *Complete interpolation of matrix versions of Heron and Heinz means*, Math. Inequal. Appl., 16 (2013), 93-99.

[2] R. Bhatia, *Matrix analysis*, Springer, New York, 1997.

[3] H. Cao, J. Wu, *Unitarily invariant norm inequalities involving Heron and Heinz means*, J. Inequal. Appl., (2014), Article ID 288.

[4] R. A. Horn, R. Mathias, *Cauchy-Schwarz inequalities associated with positive semidefinite matrices*, Linear Algebra Appl., 142 (1990), 63-82.

[5] R. A. Horn, R. Mathias, *An analog of the Cauchy-Schwarz inequality for Hadamard products and unitarily invariant norms*, SIAM J. Matrix Anal. Appl., 11 (1990), 481-498.

[6] R. Bhatia, C. Davis, *A Cauchy-Schwarz inequality for operators with applications*, Linear Algebra Appl., 223/224 (1995), 119-129.

[7] F. Hiai, X. Zhan, *Inequalities involving unitarily invariant norms and operator monotone functions*, Linear Algebra Appl., 341 (2002), 151-169.

[8] X. Hu, *Some inequalities for unitarily invariant norms*, J. Math. Inequal., 6 (2012), 615-623.

[9] L. Zou, *Unification of the arithmetic-geometric mean and Hölder inequalities for unitarily invariant norms*, Linear Algebra Appl., 562 (2019), 154-162.

[10] A. Al-Natoor, S. Benzamiab, F. Kittaneh, *Unitarily invariant norm inequalities for positive semidefinite matrices*, Linear Algebra Appl., 633 (2022), 303-315.

[11] X. Jin, F. Zhang, J. Xu, *A matrix inequality for unitarily invariant norms*, J. Math. Inequal., 16 (2022), 471-475.

[12] W. Liu, X. Hu, J. Shi, *On some inequalities related to Heinz means for unitarily invariant norms*, J. Math. Inequal., 16 (2022), 1123-1128.

[13] S. Wang, L. Zou, Y. Jiang, *Some inequalities for unitarily invariant norms of matrices*, J. Inequal. Appl., (2011), Article ID 10.

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