L-filters and TL-filters in IL-algebras

Şerife Yılmaz*
Department of Mathematics
Faculty of Science
Karadeniz Technical University
Trabzon
Turkey
serifeyilmaz@ktu.edu.tr

Huriye Betül Doğru
Department of Mathematics
Graduate School of Natural and Applied Science
Karadeniz Technical University
Trabzon
Turkey
huriyebetuldogru@gmail.com

Hashem Bordbar
Center for Information Technologies and Applied Mathematics
University of Nova Gorica
Slovenia
hashem.bordbar@ung.si

Abstract. In this paper, we introduce the concepts of L-filter and TL-filter as two different generalizations of the notion of fuzzy filter in IL-algebras. We investigate some properties with respect to these concepts. We study the relationship between L-filters and TL-filters. We give some characterizations for filters of IL-algebras by using L-filters and TL-filters. We present additional conditions so that the notions of L-filter and TL-filter coincide in an IL-algebra.

Keywords: filter, IL-algebra, L-filter, TL-filter.
MSC 2020: 03G25, 06F35, 08A72.

1. Introduction

BL-algebras were proposed by Hajek [6] in 1998 as algebraic structures formed by left-continuous triangular norms on the unit interval $[0, 1]$ and the residuation operations of these triangular norms. MV-algebras, Gödel algebras are some of the important classes of BL-algebras. Filter theory plays an important role in studying these algebraic structures and their associated logics [15]. The various filters in these algebraic structures correspond to a different class of provable

∗. Corresponding author

formulas in logics related to these algebraic structures. Filters have various applications in logic and topology. After Hajek [6] introduced the concept of filters in BL-algebra, many authors have studied different types of filters in BL-algebras. For example, Haveshki et al. [7] introduced various filter models in BL-algebras such as implicative filter, positive implicative filter, fantastic filter and examined their relationship with other filters in BL-algebras.

Fuzzy logic has been introduced by Zadeh [17]. In fuzzy logic, the real unit interval is used for modelling the set of truth values and minimum is taken for a conjunction connective. But in modern fuzzy logic, more generally, a bounded lattice takes place instead of the real unit interval and t-norms are extensively used as logical conjunction [1]. After fuzzy logic has been presented, the concept of classical mathematics has been reconsidered and fuzzy logic has been applied to the classical algebraic structures. Many authors have studied on fuzzifying different types of filters on several various algebras in recent years [3, 9, 10]. Also, Liu and Li [13] have introduced the notion of fuzzy filter and fuzzy prime filter in BL-algebras and obtained some of their properties. Then, Khorami and Saeid [11] have introduced the concept of TL-filter as a generalization of the concept of fuzzy filter of BL-algebras and investigated some basic properties of TL-filters. They have also presented a method for calculating TL-filters produced by L-subsets. It should be noted that when $L = [0, 1]$ and $T = \wedge$, the concepts of TL-filter and fuzzy filter coincide in BL-algebras.

IL-algebras, which is a generalization of BL-algebras, were introduced by Troelstra [16] in 1992 as the algebraic equivalent of intuitionistic linear logic. Islam et al. [8] presented a different definition for the concept of an IL-algebra than Troelstra's, and introduced the concepts of filter and fuzzy filter in IL-algebras. They studied related properties of these filters. They also presented three concepts of fuzzy prime filters and obtained some results correlating them with each other.

The motivation of this paper is mainly to generalize the notion of fuzzy filter in IL-algebras in a useful way. In this study, we introduce the concepts of L-filters ve TL-filters in IL-algebras, which are two different generalizations of the concept of fuzzy filter of IL-algebras, introduced by Islam et al. [8]. We investigate their related properties. We show that every L-filter is a TL-filter in an IL-algebra. We also prove some other theorems that determine the relationship between these filters.

2. Preliminaries

In this section, we give some known notions and results which will be used throughout this article.

Definition 2.1 ([2]). *Let $(L, \leq)$ be a partially ordered set. If any pair of elements x, y has an infimum and supremum, denoted by $\wedge$ and $\vee$, respectively, then $(L, \leq)$ is said to be a lattice. A lattice L is said to be complete if $\wedge S$ and $\vee S$ exist for any $S \subseteq L$. Obviously, a complete lattice has the least element and* *the greatest element denoted by* 0 *and* 1*, respectively. A lattice* L *is said to be a chain if* $x \leq y$ *or* $y \leq x$ *for any* $x, y \in L$*.*

Definition 2.2 ([14])**.** *Let* X *be a non-empty set and* L *be a complete lattice. A function from* X *to* L *is called an* L*-subset of* X *(i.e.* $\xi : X \to L$*). The set of all* L*-subsets of* X *is called the* L*-power set of* X *and is denoted by* L^X*. Also, when* L *is* $[0,1]$*,* L*-subsets of* X *are called fuzzy subsets of* X*. The set of all fuzzy subsets of* X *is called the fuzzy power set of* X *and denoted by* $[0,1]^X$*.*

Definition 2.3 ([8])**.** *Let* J *be a non-empty set. An* IL*-algebra is an algebraic system* $J = (J, \wedge, \vee, \perp, \to, *, e)$ *which satisfies the following conditions:*

i) $(J, \wedge, \vee, \perp)$ *is a lattice with the least element* $\perp$*.*

ii) $(J, *, e)$ *is a commutative monoid with the identity element* e*.*

iii) For any $x, y, z \in J$*,* $x * y \leq z$ *if and only if* $x \leq y \to z$ *(residuation property).*

Definition 2.4 ([8])**.** *Let* J *be an* IL*-algebra. A non-empty subset* F *of* J *is said to be a filter of* J *if the following conditions are satisfied:*

i) $e \in F$*.*

ii) If $x, y \in F$*, then* $x * y \in F$ *and* $x \wedge y \in F$*.*

iii) If $x \in F$ *and* $x \leq y$*, then* $y \in F$*.*

Theorem 2.1 ([8])**.** *In every* IL*-algebra* J*, the following results hold for all* $x, y, z, w \in J$*:*

i) $(x \to y) * (y \to z) \leq (x \to z)$*.*

ii) If $x \leq z$ *and* $y \leq w$*, then* $x * y \leq z * w$*.*

iii) $x * (x \to y) \leq y$*.*

Definition 2.5 ([5])**.** *Let* X *be a non-empty set and* A *a fixed subset of* X*. For any* x *in* X*, we put*

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

Then, χ_A *is a function from* X *into* $\{0,1\}$*. It is called the characteristic function of* A*.*

Definition 2.6 ([8])**.** *Let* J *be an* IL*-algebra,* L *be a complete lattice and* ξ *be an* L*-subset of* J*. Then, for each* $t \in L$*, the set* $\xi_t = \{x \in J \mid \xi(x) \geq t\}$ *is called a level subset of* ξ*.*

Definition 2.7 ([8])**.** *Let J be an IL-algebra and ξ be a fuzzy subset of J. ξ is said to be a fuzzy filter of J if ξ_t is either an empty set or a filter of J, for all $t \in [0,1]$.*

Definition 2.8 ([12])**.** *Let L be a complete lattice. A triangular norm (in short t-norm) is a binary operation T on L (i.e. $T : L \times L \longrightarrow L$) satisfying the following conditions:*

i) T is associative, i.e. for all $x, y, z \in L$,

$$T(T(x,y),z) = T(x,T(y,z)).$$

ii) T is symmetric, i.e. for all $x, y \in L$,

$$T(x,y) = T(y,x).$$

iii) T is monotone, i.e. for all $x, y, z, \in L$, if $x \leq z$, then

$$T(x,y) \leq T(z,y).$$

iv) There is a neutral element $1 \in L$ such that $T(1,x) = x$ for all $x \in L$.

Conditions (iii) and (iv) imply that for any t-norm T on L, we have

$$T(x,y) \leq x, \quad T(x,y) \leq y. \tag{1}$$

Definition 2.9 ([4])**.** *Let T_1 and T_2 be two t-norms on a complete lattice L. T_1 is called smaller than T_2 or T_2 is called greater than T_1 if $T_1(x,y) \leq T_2(x,y)$ for all $x, y \in L$. In this case, we write $T_1 \leq T_2$.*

Remark 2.1 ([4])**.** The smallest and the greatest t-norm on a complete lattice L are given by the following, respectively:

$$T_D(x,y) = \begin{cases} x \wedge y, & \text{if } x = 1 \text{ or } y = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$T_M(x,y) = x \wedge y.$$

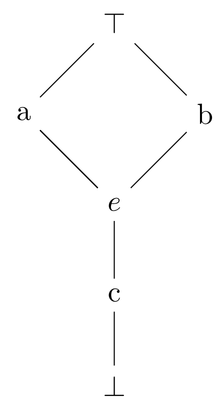
Throughout, this paper, unless otherwise stated, L denotes a complete lattice, T denotes a t-norm on L and J denotes an IL-algebra.

3. L-Filters

In this section, we introduce the notion of L-filter of an IL-algebra J and investigate some properties of L-filters. We illustrate the results with examples to better understand the concept.

Definition 3.1. *Let J be an IL-algebra, L be a complete lattice and $\xi \in L^J$. Then, ξ is said to be an L-filter of J, if for each $t \in L$, the level subset ξ_t is an empty set or a filter of J.*

Example 3.1. Let $J = \{\bot, a, b, c, e, \top\}$ be the lattice given by the following diagram.



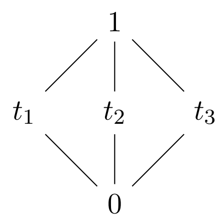
Define the binary operations $*$ and $\rightarrow$ on J by the following tables.

$*$	$\bot$	a	b	e	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\top$	a	a	$\top$
b	$\bot$	$\top$	b	b	b	$\top$
e	$\bot$	a	b	e	c	$\top$
c	$\bot$	a	b	c	c	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$

$\rightarrow$	$\bot$	a	b	e	c	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	a	$\bot$	$\bot$	$\bot$	$\top$
b	$\bot$	$\bot$	b	$\bot$	$\bot$	$\top$
e	$\bot$	a	b	e	c	$\top$
c	$\bot$	a	b	e	e	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$

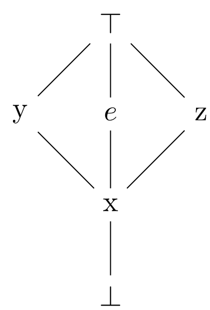
Then, $(J, \wedge, \vee, *, \rightarrow, e)$ is an IL-algebra.

Besides, let $L = \{0, t_1, t_2, t_3, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = \xi(a) = \xi(b) = 1$, $\xi(c) = t_1$ and $\xi(\perp) = 0$. Then, the level subsets of ξ for each element of L are obtained as $\xi_0 = J$, $\xi_{t_1} = \{c, e, a, b, \top\}$, $\xi_{t_2} = \xi_{t_3} = \xi_1 = \{e, a, b, \top\}$. By using Definition 2.4 and routine verification, we conclude that all level subsets of ξ are filters of J. Therefore, Definition 3.1 shows that ξ is an L-filter of J.

Example 3.2. Let $J = \{\perp, x, y, z, e, \top\}$ be the lattice given by the following diagram.

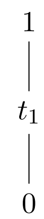


Define the binary operations $*$ and $\to$ on J by the following tables.

$*$	$\perp$	x	y	z	e	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
x	$\perp$	$\perp$	x	x	x	x
y	$\perp$	x	y	y	y	y
z	$\perp$	x	y	e	z	$\top$
e	$\perp$	x	y	z	e	$\top$
$\top$	$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\perp$	x	y	z	e	$\top$
$\perp$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
x	x	$\top$	$\top$	$\top$	$\top$	$\top$
y	$\perp$	x	$\top$	x	x	$\top$
z	$\perp$	x	y	e	z	e
e	$\perp$	x	y	z	e	$\top$
$\top$	$\perp$	x	y	x	x	$\top$

Then, $(J, \wedge, \vee, *, \to, e)$ is an IL-algebra.
Besides, let $L = \{0, t_1, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = 1$, $\xi(y) = \xi(z) = \xi(x) = t_1$ and $\xi(\bot) = 0$. Then, the level subset of ξ for t_1 is obtained as $\xi_{t_1} = \{x, y, z, e, \top\}$. Since $x \in \xi_{t_1}$ but $x * x = \bot \notin \xi_{t_1}$, then ξ_{t_1} is not a filter of J. Consequently, we get that ξ is not an L-filter of J by using Definition 3.1.

Proposition 3.1. *Let J be an IL-algebra and F be a non-empty subset of J. Then, F is a filter of J if and only if χ_F is an L-filter of J.*

Proof. Suppose that F is a filter of J and $\chi_F : J \to \{0, 1\}$ is the characteristic function of F defined by

$$\chi_F(x) = \begin{cases} 1, & x \in F \\ 0, & x \notin F \end{cases}.$$

Then, the level subsets of χ_F are $(\chi_F)_0 = J$ and $(\chi_F)_1 = F$. Thus, for each $t \in \{0, 1\}$, the level subset $(\chi_F)_t$ is a filter of J. Therefore, χ_F is an L-filter of J by Definition 3.1.

Conversely, let χ_F be an L-filter of J. Then, the level subset of χ_F for $t = 1$ is

$$(\chi_F)_1 = \{x \in J : \chi_F(x) \geq 1\} = \{x \in J : \chi_F(x) = 1\} = F.$$

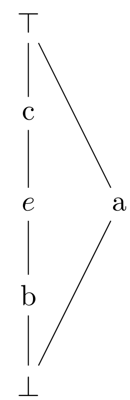
Since F is a non-empty subset of J, using Definition 3.1, we conclude that F is a filter of J. □

Proposition 3.2. *Let ξ be an L-filter of J and x, y be two arbitrary elements of J. If $x \leq y$, then $\xi(x) \leq \xi(y)$.*

Proof. Suppose that $x, y \in J$ and $x \leq y$. Then, $x \wedge y = x$ and $\xi(x \wedge y) = \xi(x)$. Let $\xi(x) = t$, for $t \in L$. Then, clearly, we have that $x \wedge y, x \in \xi_t$. Since ξ is an L-filter of J, by Definition 3.1, we conclude that the level subset ξ_t for $t \in L$ is a filter of J. Since $x \wedge y \leq y$, we have that $y \in \xi_t$ by Definition 2.4 (iii). Therefore, $\xi(x) = t \leq \xi(y)$. □

The following example shows that the converse of Proposition 3.2 does not hold.

Example 3.3. Let $J = \{\bot, a, b, c, e, \top\}$ be the complete lattice given with the following Hasse diagram.



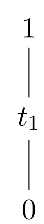
Define the binary operations $*$ and $\rightarrow$ on J as follows.

$*$	$\bot$	a	b	c	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	$\bot$	a	a	a
b	$\bot$	$\bot$	b	b	b	b
c	$\bot$	a	b	$\top$	c	$\top$
e	$\bot$	a	b	c	e	$\top$
$\top$	$\bot$	a	b	$\top$	$\top$	$\top$

$\rightarrow$	$\bot$	a	b	c	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	b	$\top$	b	b	b	$\top$
b	a	a	b	$\top$	$\top$	$\top$
c	$\bot$	a	b	e	b	$\top$
e	$\bot$	a	b	c	e	$\top$
$\top$	$\bot$	a	b	b	b	$\top$

Then, $(J, \wedge, \vee, *, \rightarrow, e)$ is an IL-algebra.

Besides, let $L = \{0, t_1, 1\}$ be the complete lattice with the following diagram.



Consider the L-subset of J, namely ξ, defined by $\xi(\top) = \xi(e) = \xi(z) = 1$, $\xi(y) = \xi(x) = \xi(\bot) = 0$. It is clear that ξ is an L-filter of J. We have that $\xi(x) \leq \xi(e)$, but $x \nleq e$.

Lemma 3.1. *Suppose that $\xi \in L^J$. If ξ is an L-filter of J, then for each $x \in J$,*

$$\xi(x) \leq \xi(e),$$

where e is the identity element of J.

Proof. Let ξ be an L-filter of J. Then, for any $t \in L$, the level subset ξ_t is a filter of J. Assume that for an arbitrary element $x \in J$, $\xi(x) = t_0$, where $t_0 \in L$. Then, clearly, we get that $x \in \xi_{t_0}$. Since ξ_{t_0} is a filter of J, then $e \in \xi_{t_0}$. Therefore, $\xi(e) \geq t_0 = \xi(x)$. □

Theorem 3.1. *Let ξ be an L-filter of J and $x, y \in J$. If $\xi(x \to y) = \xi(e)$, then $\xi(x) \leq \xi(y)$.*

Proof. Suppose that ξ is an L-filter of J. Then, the level subset ξ_t for each $t \in L$ is a filter of J by Definition 3.1. Let $\xi(x \to y) = \xi(e)$ and $\xi(x) = t$, $t \in L$. Then, clearly, we have that $x \in \xi_t$. On the other hand, since ξ_t is a filter of J, we have that $e \in \xi_t$. Hence we obtain that $t \leq \xi(e) = \xi(x \to y)$. Then, we have that $x \to y \in \xi_t$. Using Definition 2.4 (ii), we get that $x * (x \to y) \in \xi_t$. Since we know that $x * (x \to y) \leq y$ by Theorem 2.1 (iii) and ξ_t is a filter of J, we get that $y \in \xi_t$. So $\xi(x) = t \leq \xi(y)$. □

By the following example, we show that the converse of the above theorem does not hold.

Example 3.4. Consider the L-filter ξ defined in Example 3.3. Note that $\xi(x) \leq \xi(e)$ but $\xi(x \to e) = \xi(y) = 0 \neq \xi(e)$.

4. TL-Filters

Definition 4.1. *Let $\xi \in L^J$ and T be a t-norm on L. Then, ξ is called a TL-filter of J if and only if, for any $x, y \in J$, the following conditions are satisfied.*

i) $x \leq y \Longrightarrow \xi(x) \leq \xi(y)$,

ii) $T(\xi(x), \xi(y)) \leq \xi(x \wedge y)$,

iii) $T(\xi(x), \xi(y)) \leq \xi(x * y)$,

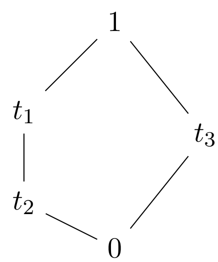
iv) $\xi(x) \leq \xi(e)$.

Theorem 4.1. *In an IL-algebra J, every L-filter is a TL-filter.*

Proof. The conditions (i) and (iv) of Definition 4.1 are clear from Proposition 3.2 and Lemma 3.1, respectively. Now, let ξ be an L-filter of J. Suppose that $T(\xi(x), \xi(y)) = t$ for $t \in L$. Using inequalities (1), we conclude that $T(\xi(x), \xi(y)) \leq \xi(x) \wedge \xi(y)$. Hence $t \leq \xi(x)$ and $t \leq \xi(y)$. Then, clearly, we have that $x, y \in \xi_t$. Since ξ is an L-filter of J, by Definition 3.1, we know that the level subset ξ_t, for any $t \in L$, is a filter of J. Thus, by Definition 2.4 (ii), we have that $x \wedge y \in \xi_t$ and $x * y \in \xi_t$. Thus, $\xi(x \wedge y) \geq t$ and $\xi(x * y) \geq t$. Therefore, $\xi(x \wedge y) \geq t = T(\xi(x), \xi(y))$ and $\xi(x * y) \geq t = T(\xi(x), \xi(y))$. □

The following example shows that the converse of Theorem 4.1 is not true in general.

Example 4.1. Let J be the IL-algebra given in Example 3.3 and $L=\{0, t_1, t_2, t_3, 1\}$ be the complete lattice given with the following diagram.



Now, define the function $\xi : J \to L$ by $\xi(e) = \xi(c) = \xi(\top) = t_1$, $\xi(b) = \xi(a) = t_2$ and $\xi(\bot) = 0$. Since $a, b \in \xi_{t_2}$ but $a \wedge b = \bot \notin \xi_{t_2}$, the level subset ξ_{t_2} is not a filter of J. So, we get that ξ is not an L-filter of J.

Consider the smallest t-norm T_D on L in Remark 2.1 as follows.

$$T_D(x, y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then, we can easily verify that ξ is a TL-filter of J.

Theorem 4.2. *Let F be a non-empty subset of J. Then, F is a filter of J if and only if χ_F is a TL-filter of J.*

Proof. Suppose that F is a filter of J. By Proposition 3.1, we have that χ_F is an L-filter of J. Since every L-filter is a TL-filter in an IL-algebra by Theorem 4.1, χ_F is a TL-filter of J.

Conversely, let χ_F be a TL-filter of J. Since $F \subseteq J$ is a non-empty set, then there exists $x \in F$. So, $\chi_F(x) = 1$. Since χ_F is a TL-filter of J, then by Definition 4.1 (iv), $\chi_F(x) \leq \chi_F(e)$. Thus, $1 \leq \chi_F(e)$ which means that $\chi_F(e) = 1$ and whence $e \in F$.

Now, suppose that x, y are two arbitrary elements of F. Then, $\chi_F(x) = \chi_F(y) = 1$ and

$$T(\chi_F(x), \chi_F(y)) = T(1, 1) = 1.$$

Since χ_F is a TL-filter of J, then by Definition 4.1, we have that $1 = T(\chi_F(x), \chi_F(y)) \leq \chi_F(x * y)$. Hence $\chi_F(x * y) = 1$ and therefore $x * y \in F$. Similarly, we can prove that $x \wedge y \in F$ for any $x, y \in F$. Finally, suppose that $x \leq y$ and $x \in F$. Then, $\chi_F(x) = 1$. Because of that χ_F is a TL-filter of J, we get that $\chi_F(x) \leq \chi_F(y)$ which means that $1 = \chi_F(y)$ and $y \in F$. Therefore, F is a filter of J. □

Proposition 4.1. *If ξ is a TL-filter of J, then for any $x, y \in J$, we have that $T(\xi(x), \xi(x \to y)) \leq \xi(y)$.*

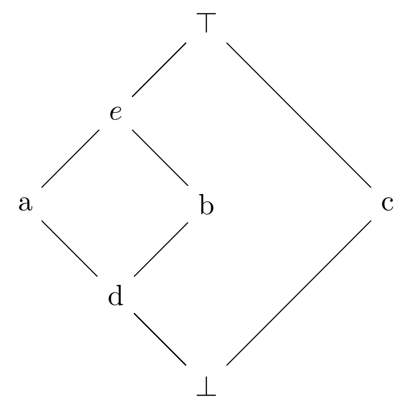
Proof. By Theorem 2.1 (iii), we know that $x * (x \to y) \leq y$. Since ξ is a TL-filter of J, we have that $\xi(x * (x \to y)) \leq \xi(y)$. Therefore, we obtain that

$$T(\xi(x), \xi(x \to y)) \leq \xi(x * (x \to y)) \leq \xi(y).$$

□

In an IL-algebra, some results concerning L-filters are not true for TL-filters. The following example shows that Theorem 3.1 does not hold if ξ is a TL-filter of J.

Example 4.2. Let $J = \{\bot, a, b, c, d, e, \top\}$ be the complete lattice given with the following Hasse diagram.



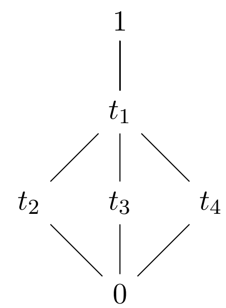
Define the binary operations $*$ and $\to$ on J as follows.

$*$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	d	c	d	a	$\top$
b	$\bot$	d	b	c	d	b	$\top$
c	$\bot$	c	c	d	c	c	$\top$
d	$\bot$	d	d	c	d	d	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

$\to$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	$\bot$	e	b	c	b	e	$\top$
b	$\bot$	a	e	c	a	e	$\top$
c	$\bot$	c	c	e	c	c	$\top$
d	$\bot$	e	e	c	e	e	$\top$
e	$\bot$	a	b	c	d	e	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$

Then, $(J, \wedge, \vee, \bot \to, *, \mathrm{e})$ is an IL-algebra.

Besides, let $L = \{0, t_1, t_2, t_3, t_4, 1\}$ be the complete lattice given in the following diagram.



Now, consider the function $\xi : J \to L$ defined by $\xi(1) = \xi(\top) = \xi(a) = 1$, $\xi(b) = t_2$, $\xi(c) = t_3$, $\xi(d) = \xi(\bot) = 0$ and the smallest t-norm T_D on L in Remark 2.1 as follows.

$$T_W(x, y) = \begin{cases} y, & x = 1 \\ x, & y = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By routine verification, ξ is a TL-filter of J. Note that

$$\xi(b \to d) = \xi(a) = 1$$

but $\xi(b) = t_2 \not\leq \xi(d) = 0$.

Proposition 4.2. *Let ξ be a TL-filter of J. Then, for all $x, y, z \in J$, we have the following statements.*

i) If $x \wedge y \leq z$, then $T(\xi(x), \xi(y)) \leq \xi(z)$.

ii) If $x * y \leq z$, then $T(\xi(x), \xi(y)) \leq \xi(z)$.

Proof. The proof easily comes from by Definition 4.1 (i) and (ii). □

Proposition 4.3. *Let ξ be a TL-filter of J. Then, for all $x, y, z \in J$, we have the following inequality:*

$$T(\xi(x \to y), \xi(y \to z)) \leq \xi(x \to z).$$

Proof. We know from Theorem 2.1 (i) that $(x \to y) * (y \to z) \leq (x \to z)$. Using Definition 4.1 (i), we obtain that $\xi((x \to y) * (y \to z)) \leq \xi(x \to z)$. Therefore, by using Definition 4.1 (iii), we get that $T(\xi(x \to y), \xi(y \to z)) \leq \xi(x \to z)$. □

Theorem 4.3. *Let ξ be a TL-filter of J. If J is a chain and $T = T_M$, then ξ is an L-filter of J.*

Proof. Suppose that ξ is a TL-filter of J. To show that ξ is an L-filter of J, we should prove that for any $t \in L$, the level subset ξ_t is an empty set or a filter of J. Let ξ_t is a non-empty set. Then, there exists $x \in \xi_t$ and so $\xi(x) \geq t$. Since ξ is a TL-filter, we have that $\xi(x) \leq \xi(e)$. So we obtain that $\xi(e) \geq t$ and whence $e \in \xi_t$. Let $x \in \xi_t$ and $x \leq y$. Then, by Definition 4.1 (i), we get that $\xi(y) \geq \xi(x)$. Therefore, $\xi(y) \geq t$ and $y \in \xi_t$. Since $T(\xi(x), \xi(y)) \leq \xi(x \wedge y)$ by Definition 4.1 (ii) and $T = T_M$, we have that $\xi(x) \wedge \xi(y)) \leq \xi(x \wedge y)$. On the other hand, since J is a chain, we have $x \geq y$ or $y \geq x$. So $\xi(x) \geq \xi(y)$ or $\xi(y) \geq \xi(x)$. Hence $\xi(x) \wedge \xi(y) = \xi(x)$ or $\xi(x) \wedge \xi(y) = \xi(y)$. In both cases, we get that $\xi(x \wedge y) \geq t$. So $x \wedge y \in \xi_t$. By a similar argument and Definition 4.1 (iii), we get that $x * y \in \xi_t$. Consequently, ξ_t is a filter of J. It proves that ξ is an L-filter of J. □

5. Conclusions

We have defined the notions of L-filter and TL-filter in IL-algebras and have studied their properties. We have investigated the relationship between L-filters and TL-filters. We have characterized filters by using L-filters and TL-filters. For future work, characterizations and calculations of these filters may be investigated. Also, other kinds of filters such as implicative TL-filters, positive implicative TL-filters can be studied in IL-algebras.

References

[1] C. Alsina, E. Trillas, L. Valverde, *On non-distributive logical connectives for fuzzy set theory*, BUSEFAL, 3 (1980), 18–29.

[2] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., Providence, Rhode Island, 1940.

[3] B. Davvaz, O. Kazancı, *A new kind of $(\epsilon, \epsilon \vee q)$-fuzzy sublattice (ideal, filter) of a lattice*, Int. J. Fuzzy Syst., 13 (2011), 55-63.

[4] B. De Baets, R. Mesiar, *Triangular norms on product lattices*, Fuzzy Sets Syst., 104 (1999), 61-75.

[5] A. K. Feyzioğlu, *A course on algebra*, Boğaziçi University Printing Office, 1990.

[6] P. Hajek, *Metamathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.

[7] M. Haveshki, A. Borumand Saeid, E. Eslami, *Some type of filters of BL-algebras*, Soft Comput., 10 (2006), 657–664.

[8] S. Islam, A. Sanyal, J. Sen, *Fuzzy filters of IL-algebras*, Soft Comput., 26 (2022), 7017-7027.

[9] Y. B. Jun, Y. Xu, X. H. Zhang, *Fuzzy filters of MTL-algebras*, Inform. Sci., 175 (2005), 120-138.

[10] A. Kadji, C. Lele, M. Tonga, *Fuzzy prime and maximal filters of residuated lattices*, Soft Comput., 2016, 1-10.

[11] R. T. Khorami, A. B. Saeid, *New representation for filters of BL-algebras*, J. Intel. Fuzzy Syst., 23 (2012), 225-235.

[12] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Springer Science and Business Media, Dordrecht, 2013.

[13] L.Z. Liu, K.T. Li, *Fuzzy filters of BL-algebras*, Inform. Sci., 173 (2005), 141–154.

[14] J. N. Mordeson, D. S. Malik, *Fuzzy commutative algebra*, World Scientific, 1998.

[15] H. Rasiowa, *An algebraic approach to non-classical logics*, North Holland, Amsterdam, 1974.

[16] A. S. Troelstra, *Lectures on linear logic*, Center for the Study of Language and Information Publications, Stanford, 1992.

[17] L.A. Zadeh, *Fuzzy sets*, Information Control, 8 (1965), 338-353.

Accepted: October 21, 2024